

Blind Null-Space Learning for MIMO Underlay Cognitive Radio Networks

Yair Noam and Andrea J. Goldsmith, *Fellow, IEEE*

Abstract

This paper proposes a blind technique for MIMO cognitive radio Secondary Users (SU) to transmit in the same band simultaneously with a Primary User (PU) under a maximum interference constraint. In the proposed technique, the SU is able to meet the interference constraint of the PU without explicitly estimating the interference channel matrix to the PU and without burdening the PU with any interaction with the SU. The only condition required of the PU is that for a short time interval it uses a power control scheme such that its transmitted power is a monotonic function of the interference inflicted by the SU. During this time interval, the SU iteratively modifies the spatial orientation of its transmitted signal and measures the effect of this modification on the PU's total transmit power. The entire process is based on energy measurements which is very desirable from an implementation point of view.

I. INTRODUCTION

The emergence of Multiple Input Multiple Output (MIMO) communication opens new directions and possibilities for Cognitive Radio (CR) networks [1–3]. In particular, in underlay CR networks, MIMO technology enables the SU to transmit a significant amount of power simultaneously in the same band with the PU without interfering with him if the SU utilizes separate spatial dimensions than the PU. This spatial separation requires that the interference channel from the SU to the PU be known to the SU. Thus, acquiring this knowledge, or operating without it, is a major issue of active research [4–8]. We consider MIMO primary and secondary systems defined as follows: we assume a flat-fading MIMO channel with one primary and

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multiple SUs. Let $\mathbf{H}_{1j} \in \mathbb{C}^{n_1 \times n_j}$ be the channel matrix between user j 's transmitter and the PU's receiver. In the underlay CR paradigm, SUs are constrained not to exceed a maximum interference level at the PU, i.e.

$$\|\mathbf{H}_{1j}\mathbf{x}_j(t)\|^2 \leq \eta, \quad \forall j \neq 1, \quad (1)$$

where \mathbf{x}_j is SU j 's ($j > 1$) transmit signal and $\eta > 0$ is the maximum interference constraint. If $\eta = 0$, the SUs are strictly constrained to transmit only within the null space of the matrix \mathbf{H}_{1j} .

The optimal power allocation for the case of a single SU who knows the matrix \mathbf{H}_{21} in addition to its own Channel State Information (CSI) was derived by Zahng and Liang [1]. For the case of multiple SUs, Scutari et al. [3] formulated a competitive game between the secondary users. Assuming that the interference matrix to the PU is known by each SU, they derived conditions for the existence and uniqueness of a Nash Equilibrium point to the game. Zhang et al. [5] were the first to take into consideration the fact that the interference matrix \mathbf{H}_{12} may not be perfectly known (but is partially known) to the SU. They proposed Robust Beamforming to assure compliance with the interference constraint of the PU while maximizing the SU's throughput. Other work for the case of an unknown interference channel with known probability distribution is due to Zhang and So [6], who optimized the SU throughput under a constraint on the maximum probability that the interference to the PU is above a threshold.

The underlay concept of CR in general and MIMO CR in particular is that the SU must be able to mitigate the interference to the PU blindly without any cooperation. Yi [8] proposed a solution in the case where the SU learns the channel matrix based on channel reciprocity between the PU where the SU listens to the PU transmitted signal and estimates \mathbf{H}_{12} 's null space from the signal's second order statistics. This work was enhanced by Chen et. al. [7]. Both works require channel reciprocity and therefore are restricted to a PU that uses Time Division Duplexing (TDD). Once the SU obtains the null space of \mathbf{H}_{12} , it does not interfere with the PU as long as his signal occupies that null space.

Other than in the channel reciprocity case, obtaining the value of \mathbf{H}_{1j} by the SUs (i.e. the interference channel to the PU) requires cooperation from the PU in the estimation phase, e.g. where the SU transmits a training sequence, from which the PU estimates \mathbf{H}_{1j} and feeds it back to the SU. Cooperation of this nature increases system complexity overhead, since it requires a handshake between both systems and in addition, the PU needs to be synchronized to the SU's

training sequence. This is one of the major technical obstacles that prevents underlay CRs from being widespread.

The objective of this work is to design a simple procedure such that MIMO underlay SU can meet the interference constraint to the PU without explicitly estimating the matrix \mathbf{H}_{1j} and without burdening the PU with any handshaking, estimation or synchronization associated with the SUs. In the proposed scheme (see Fig. 1) the PU is not cooperating at all with the SU and operates as if it is the only system in the medium (as current PUs operate today). The only condition required is that for some short time interval (that may be much shorter than the entire learning process), the PU will be using a power control scheme such that its transmitted power is a monotonic function of the interference inflicted by the SU. Under this condition, we propose a learning algorithm in which the SU is gradually reducing the interference to the PU by iteratively modifying the spatial orientation of its transmitted signal and measuring the effect of this modification on the PU's total transmit power. The entire process is based on energy measurements and on detecting energy variations. Therefore, it does not require any handshake or synchronization between the PU and the SU.

The paper is organized as follows: Section II provides the system description and some notation. Section III presents a blind approach for realizing the Cyclic Jacobi technique for calculating the Eigenvalue Decomposition (EVD) of an unobserved matrix; this will be the building block of the blind null space learning algorithm presented in Section IV and analyzed in Section V. Simulations and Conclusions are presented in Sections VI and VII, respectively.

II. PROBLEM FORMULATION

Consider a flat fading MIMO interference channel with a single PU and a single SU without interference cancellation, i.e. each system treats the other system's signal as noise. The PU's received signal is given by

$$\mathbf{y}_1(t) = \mathbf{H}_{11}\mathbf{x}_1(t) + \mathbf{H}_{12}\mathbf{x}_2(t) + \mathbf{v}_1(t), \quad t \in \mathbb{N} \quad (2)$$

where $\mathbf{v}_1(t)$ is a zero mean stationary noise. In this paper all vectors are column vectors. Let \mathbf{A} be an $l \times m$ complex matrix, then, its null space is defined as $\mathcal{N}(\mathbf{A}) = \{\mathbf{y} \in \mathbb{C}^m : \mathbf{A}\mathbf{y} = \mathbf{0}\}$ where $\mathbf{0} = [0, \dots, 0]^T$. For simplicity \mathbf{H}_{12} will be denoted by \mathbf{H} and the matrix $\mathbf{H}^*\mathbf{H}$ will be denoted by \mathbf{G} . The time line \mathbb{N} will be divided into N -length intervals referred to as transmission

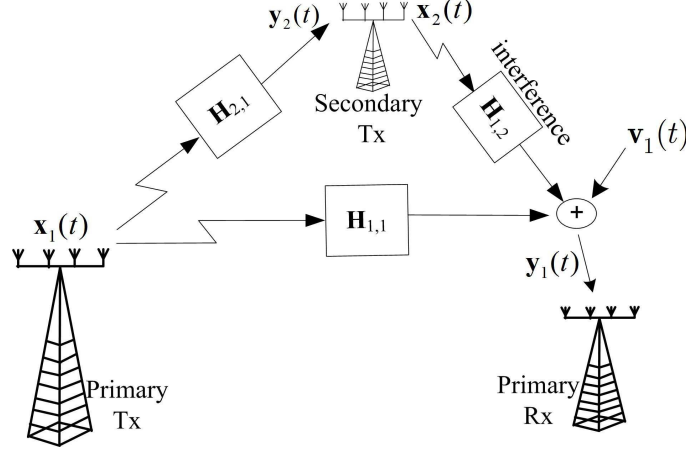


Fig. 1. The addressed cognitive radio scheme. The matrix $\mathbf{H}_{1,2}$ is unknown to the secondary transmitter and $\mathbf{v}_1(t)$ is a stationary noise (which may include stationary interference). The interference from the SU, $\mathbf{H}_{1,2}\mathbf{x}_2(t)$, is treated as noise, i.e. no interference cancellation is performed. The SU measures the energy variations in $\mathbf{y}_2(t)$ according to which it varies its transmission scheme until the interference to the PU becomes sufficiently small such that it does not affect $\mathbf{y}_2(t)$.

cycles where, for each cycle, the SU's signal will be constant (this is required only during the learning process), i.e.

$$\begin{aligned} \mathbf{x}_2((n-1)N + N') &= \mathbf{x}_2((n-1)N + 1) \\ &= \dots = \mathbf{x}_2(Nn + N' - 1) \triangleq \tilde{\mathbf{x}}(n), \end{aligned} \quad (3)$$

where the time interval $nN < t \leq nN + N' - 1$ ($N' \ll N$) is the snapshot in which the SU measures a function

$$q(n) = \frac{1}{N'} \sum_{t=Nn}^{Nn+N'-1} \|\mathbf{y}_2(t)\|^2 \quad (4)$$

where \mathbf{y}_2 is the observed signal at the secondary transmitter that includes the primary system's transmitted signal (see Figure 1). The SU's objective is to learn $\mathcal{N}(\mathbf{H})$ from $\{\tilde{\mathbf{x}}(n), q(n)\}_{n \in \mathbb{N}}$. This learning process is carried out in learning stages where each stage consists of K transmission cycles. We will index each learning stage by k . The indexing method is depicted in Figure 2. During the learning process, the SU varies the interference to the PU by transmitting a different interfering signal $\tilde{\mathbf{x}}(n)$. The secondary transmitter measures $\mathbf{y}_2(t)$ from which it extracts $q(n)$ in order to monitor the PU's transmitted power, i.e. $\frac{1}{N'} \sum_{t=nN}^{Nn+N'-1} \|\mathbf{x}_1(t)\|^2$. Each transmission cycle, N , corresponds to the PU's power control cycle, i.e. to the time interval between two

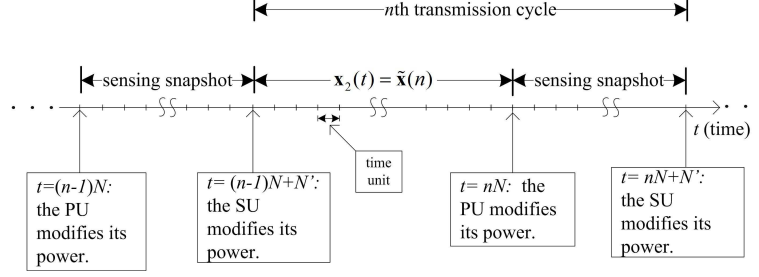


Fig. 2. The time indexing that is used in this paper. t indexes the basic time unit (pulse time) where N time units constitute a transmission cycle that is indexed by n . Furthermore, K transmission cycles constitute a learning phase (this is not illustrated in this figure).

consecutive power adaptations made by the PU. In fact the actual $q(n)$ is not important as long as it satisfies the following condition.

Assumption 1: Let $k \in \mathbb{N}$, then for every $K(k-1) \leq n < kK$ the function $q(n)$ satisfies $q(n) = f_k(\|\mathbf{H}\tilde{\mathbf{x}}(n)\|^2)$ where $f_k : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is a sequence of strictly monotonous continuous increasing (decreasing) functions.

Without loss of generality, we assume that f_k is a sequence of monotone increasing functions. The most important trait (that will be used in this paper) that follows from Assumption 1 is that for every $K(k-1) \leq n, m < kK$, $f_k(\|\mathbf{H}\tilde{\mathbf{x}}(n)\|^2) \geq f_k(\|\mathbf{H}\tilde{\mathbf{x}}(m)\|^2)$ implies that $\|\mathbf{H}\tilde{\mathbf{x}}(n)\|^2 \geq \|\mathbf{H}\tilde{\mathbf{x}}(m)\|^2$. The problem is illustrated in Figure 3. From a system point of view, Assumption 1 means that between two consecutive transmission cycles the primary transmitter's power can be modified only due to variations in the interference level from the SU that is at the learning process and not in a steady state.

In the following we provide an example for conditions under which Assumption 1 is satisfied. To simplify the exposition, we replace the function q in (4) with

$$q(n) = \frac{1}{N'} \sum_{t=nN}^{Nn+N'-1} \mathbb{E}\{\|\mathbf{y}_2(t)\|^2\} \quad (5)$$

In doing so, we ignore the measurement noise at the secondary transmitter. Such $q(n)$ satisfies Assumption 1 if for example

- 1) The PU's signal has the following form: $\mathbf{x}_1(t) = \sqrt{p_1(n)}\mathbf{P}_1\mathbf{s}(t)$ where \mathbf{P}_1 is a constant pre-coding matrix, $\mathbb{E}\{\mathbf{s}(t)\mathbf{s}^*(t)\} = \mathbf{I}$, and $p_1(n)$ is the PU's power level at the n^{th} cycle.

This is satisfied for example if the PU is using beamforming (in this case $\mathbf{s}(t)$ is a scalar

and \mathbf{P}_1 is a rank 1 matrix), or using a uniform power allocation (in this case $\mathbf{P}_1 = \mathbf{I}$) or if the PU has a single antenna at the transmitter.

- 2) If in addition to condition 1 the PU has an SNR constraint at the receiver or a constant rate constraint. Then $p_1(n)$ will be a monotonic increasing function of its total noise plus interference.
- 3) Assume that condition 1 is satisfied and in addition the PU is using OFDMA in which the current channel is just one of the bins and assume that the PU is using the water filling rule. In that case, $p_1(n)$ will be a decreasing function of the interference plus noise.

Under such conditions and assuming that \mathbf{H}_{21} is unchanged between two transmission cycles, we have.

$$\begin{aligned}
 q_p(n) &= \mathbb{E}\{\|\mathbf{y}_2(t)\|^2\} \\
 &= p_1(n)\mathbb{E}\{\mathbf{H}_{21}\mathbf{P}_1\mathbf{s}(t)\mathbf{s}(t)^*\mathbf{H}_{21}\mathbf{P}_1^*\} + \mathbb{E}\{\|\mathbf{v}_2(t)\|^2\} \\
 &= p_1(n)\text{Tr}\{\mathbf{H}_{21}\mathbf{P}_1\mathbf{H}_{21}\mathbf{P}_1^*\} + \mathbb{E}\{\|\mathbf{v}_2(t)\|^2\} \\
 &= a_1p_1(n) + a_2
 \end{aligned} \tag{6}$$

The learning process is carried out as follows: At the first cycle ($n = 1$), the SU transmits a low-power signal $\tilde{\mathbf{x}}(1)$, such that the interference constraint (1) is satisfied¹ and measures the PU's transmit energy $q(1)$. At the next cycle, the SU transmits $\tilde{\mathbf{x}}_2(2)$ and measures $q(2)$ and so on. Section IV describes algorithms for the SU to learn the null space of the interference channel matrix \mathbf{H} based on these measurements, i.e. to approximate $\mathcal{N}(\mathbf{H})$ from $\{\tilde{\mathbf{x}}(n), q(n)\}_{n=1}^T$ where the accuracy can be arbitrary small for sufficiently large T . The algorithm is not limited to networks with a single SU; it is also valid for networks with multiple SUs as long as only one system modifies its power allocation during its learning process. This fact enables a new SU to join the channel in which multiple SUs coexist with the PU in a steady-state (i.e. a case where each SU meets its interference constraint given in (1)).

III. BLIND JACOBI DIAGONALIZATION

In this section we present a blind approach for realizing the Jacobi technique for calculating the EVD of an unobserved Hermitian matrix \mathbf{G} assuming that only $S(\mathbf{G}, \mathbf{x}) = \mathbf{x}^*\mathbf{G}\mathbf{x}$ is observed.

¹ This can be obtained by transmitting a very low-power signal at the first cycle. If $q_p(n)$ is not affected, the power of $\tilde{\mathbf{x}}$ can be gradually increased until $q(n)$ is affected.

This algorithm will be the building block of the blind null space learning algorithm that is presented in Section IV.

A. The Jacobi Technique

The Jacobi technique obtains the EVD of the Hermitian matrix \mathbf{G} via a series of 2-dimensional rotation that eliminates two off-diagonal elements in each phase (indexed by k). It begins with setting $\mathbf{A}_0 = \mathbf{G}$ and then performing the following rotation operations

$$\mathbf{A}_{k+1} = \mathbf{V}_k \mathbf{A}_k \mathbf{V}_k^*, k = 1, 2, \dots \quad (7)$$

where $\mathbf{V}_k = \mathbf{R}_{l,m}(\theta, \phi)$ is an $n_t \times n_t$ unitary rotation matrix whose p, q entry is given by:

$$[\mathbf{R}_{l,m}(\theta, \phi)]_{p,q} = \begin{cases} \cos(\theta) & \text{if } p = q \in \{m, l\} \\ e^{-i\phi} \sin(\theta) & \text{if } p = m \neq q = l \\ -e^{i\phi} \sin(\theta) & \text{if } p = l \neq q = m \\ 1 & \text{if } p = q \notin \{m, l\} \\ 0 & \text{otherwise} \end{cases} \quad (8)$$

For each k , the vales of θ, ϕ are chosen such that $[\mathbf{A}_{k+1}]_{l,m} = 0$, in words, θ and ϕ are chosen to zero \mathbf{A}_k 's l, m and m, l off diagonal entries. The values of l, m are chosen in step k according to a function $J : \mathbb{N} \longrightarrow \{1, \dots, n_t\} \times \{1, \dots, n_t\}$ i.e $J_k = (l_k, m_k)$. It is the choice of J_k that differs between different Jacobi techniques. For example, in the classic Jacobi technique, the off diagonal elements are chosen according to

$$(l_k, m_k) = \arg \max_{\{(l,m): l > m\}} (|[\mathbf{A}_k]_{l,m}|) \quad (9)$$

which corresponds the maximal off-diagonal entry². In the cyclic Jacobi method the rotation rule is defined as follows:

Definition 1: $J_k = (l_k, m_k)$ is a function such that $1 < l_k < n_t - 1$ and $l_k < m_k \leq n_t$ where each pair (l, m) is chosen once in each cycle. Unless otherwise stated it is assumed that $l_k \leq l_q$ if $k \leq q$ and that $m_k \leq m_q$ if $k \leq q$ and $l_k = l_q$.

For example if $n_t = 3$ then $J_1 = (1, 2)$, $J_2 = (1, 3)$, $J_3 = (2, 3)$, $J_4 = (1, 2) \dots$

² Recall that \mathbf{A} is Hermitian therefore it is sufficient to restrict $m > l$.

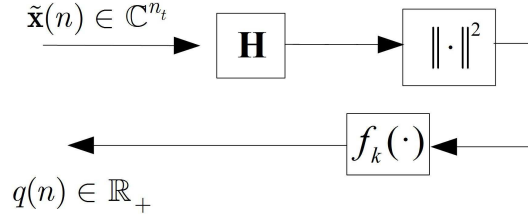


Fig. 3. Block Diagram of the Blind Null Space Learning Problem. The SU's objective is to learn the null space of \mathbf{H} by inserting a series of $\{\tilde{\mathbf{x}}(n)\}_{n \in \mathbb{N}}$ and measuring $q(n)$ as output. The only information that can be extracted is that $\|\mathbf{H}\tilde{\mathbf{x}}(n)\|^2 \geq \|\mathbf{H}\tilde{\mathbf{x}}(l)\|^2$ if $q(n) \geq q(l)$ for every $(k-1)K \leq l, n \leq kK$ where $k \in \mathbb{N}$.

B. Blind Jacobi Step

In the proposed cognitive radio scenario, if the SU wishes to perform the Jacobi step it has to do so without observing \mathbf{G} and \mathbf{A}_k but observing only $f_k(\|\mathbf{H}\mathbf{x}(n)\|^2) = f_k(\mathbf{x}^*(n)\mathbf{G}\mathbf{x}(n))$ as depicted in Figure 3. The following proposition shows how to do this:

Theorem 1: Let \mathbf{G} be an $n_t \times n_t$ Hermitian matrix, let $\mathbf{R}_{l,m}(\theta, \phi)$ be defined in Eq. (8), and let $S(\mathbf{G}, \mathbf{a}) = \mathbf{a}^* \mathbf{G} \mathbf{a}$. The values of θ and ϕ that eliminate the l, m entry of \mathbf{G} i.e.

$$[\mathbf{R}_{l,m}(\theta, \phi) \mathbf{G} \mathbf{R}_{l,m}^*(\theta, \phi)]_{l,m} = 0 \quad (10)$$

are given by

$$(\hat{\theta}, \hat{\phi}) = \arg \text{ext}_{\theta, \phi} S(\mathbf{G}, \mathbf{r}_{l,m}(\theta, \phi)) \quad (11)$$

where $\arg \text{ext}$ denotes an extreme point and $\mathbf{r}_{l,m}(\theta, \phi)$ denotes the l th column of $\mathbf{R}_{l,m}(\theta, \phi)$. Furthermore, every local minimum point of the function $S(\mathbf{G}, \mathbf{r}_{l,m}(\theta, \phi))$ is also an absolute minimum point, i.e., let $\Gamma^* = \{(\theta^*, \phi^*) = \gamma^* \in \mathbb{R}^2 : \exists \epsilon > 0, S(\mathbf{G}, \mathbf{r}_{l,m}(\gamma^*)) \leq S(\mathbf{G}, \mathbf{r}_{l,m}(\gamma)), \forall \|\gamma - \gamma^*\| < \epsilon\}$, then $S(\mathbf{G}, \mathbf{r}_{l,m}(\gamma_1)) = S(\mathbf{G}, \mathbf{r}_{l,m}(\gamma_2)), \forall \gamma_1, \gamma_2 \in \Gamma^*$. The same statement applies to local maxima.

Proof: See Appendix A.

Theorem 1 asserts that the Jacobi step can be carried out by a blind two-dimensional optimization in which every local minimum is a global minimum. This is a very important property since it is easy to identify if an optimization algorithm has converged to a local minimum. Note that Theorem 1 applies also if $S(\mathbf{G}, \mathbf{x})$ is not observed but $f_k(S(\mathbf{G}, \mathbf{x}))$ is observed instead.

C. Reducing the Two-Dimensional Optimization into Two One-Dimensional Optimizations

Although the Jacobi step can be implemented blindly, it requires a two dimensional optimization over the parameters θ and ϕ . This may be very difficult in practice since each search point is obtained via a transmission cycle. Fortunately, this two dimensional optimization can be carried out optimality by two one-dimensional optimizations as shown in following theorem:

Theorem 2: Let $S(\mathbf{G}, \mathbf{x}) = \mathbf{x}^* \mathbf{G} \mathbf{x}$ and let $\mathbf{r}_{l,m}(\theta, \phi)$ be $\mathbf{R}_{l,m}(\theta, \phi)$'s l^{th} column where $\mathbf{R}_{l,m}(\theta, \phi)$ is defined in (8). The optimal Jacobi parameters $\hat{\theta}$ and $\hat{\phi}$ in (11) can be achieved by

$$\hat{\phi} = \arg \min_{\phi \in [-\pi, \pi]} S(\mathbf{G}, \mathbf{r}_l(\pi/3, \phi)) \quad (12)$$

$$\hat{\theta} = \begin{cases} \tilde{\theta} & \text{if } -\pi/4 \leq \tilde{\theta} \leq \pi/4 \\ \tilde{\theta} - \text{sign}(\tilde{\theta})\pi/2 & \text{otherwise} \end{cases} \quad (13)$$

where $\text{sign}(\theta) = 1$ if $\theta > 0$ and -1 otherwise and

$$\tilde{\theta} = \arg \min_{\theta \in [-\pi/2, \pi/2]} S(\mathbf{G}, \mathbf{r}_{l,m}(\theta, \hat{\phi})) \quad (14)$$

Proof: See Appendix A.

Comment: Note that the rotation angle θ_k is restricted to the interval $[-\pi/4, \pi/4]$. In Section IV-B it is shown that this restriction guarantees a globally linear convergence rate and ultimately a quadratic convergence rate.

In practice, the optimizations in (12), (14) will be carried out using line searches. This is because, the function $S(\mathbf{G}, \mathbf{x})$ will not be observed³ but only $f_k(S(\mathbf{G}, \mathbf{x}))$ will be observed (see Figure 3). The complexity of the one dimensional line search in (12) can be drastically reduced if one is looking for a minimum (or maximum) and if the objective function has a single minimum (local and global). Under these conditions a binary search can be invoked. This is important since it reduces the complexity of the line search drastically from $O(1/\eta)$ to $O(\log(1/\eta))$ where η is the line search accuracy. In the sequel, it will be shown how to solve both (12) and (14) using a binary search. We begin with the optimization in (14) which can be written as

$$\tilde{\phi}_k = \arg \min_{\phi \in [-\pi, \pi]} f_k(S(\mathbf{G}, \mathbf{r}_{l,m}(\pi/3, \phi))) \quad (15)$$

³In [9] it is shown that $S(\mathbf{G}, \mathbf{x})$ is known, the problem can be simplified drastically where \mathbf{G} can be obtained precisely by a finite number of transmission cycles.

Note that

$$S(\mathbf{G}, \mathbf{r}_{l,m}(\theta, \phi)) = \cos^2(\theta) |g_{l,l}| + \sin^2(\theta) |g_{m,m}| - |g_{l,m}| \sin(2\theta) \cos(\phi + \angle g_{l,m}) \quad (16)$$

It follows that during each learning phase, that is for every $(k-1)K \leq n \leq kK$, the function $f_k(S(\mathbf{G}, \mathbf{r}_{l,m}(0.5, \phi)))$ is equivalent to

$$w_k(\phi) = f_k(A + B \cos(\phi + \angle g_{l,m})) \quad (17)$$

where f_k is a monotone function and $A \geq |B|$.

Proposition 3: Consider the function w in (17) then

- a. $w(\phi)$ is monotonic on $[\pi/2, \pi]$ and non-monotonic on $(0, \pi/2)$ if and only if

$$|w(0) - w(\pi/2)| \leq |w(\pi/2) - w(\pi)| \quad (18)$$

- b. Let $\check{\phi} \in (-\pi, \pi)$ be a minimum⁴ point of $w(\phi)$ then:

- i. Assume (18) is true, then, $\check{\phi} \in [0, \pi/2]$ if $w(\pi/2) \geq w(\pi)$ and $\check{\phi} \in [-\pi, -\pi/2]$ if $w(\pi/2) \geq w(\pi)$.
- ii. Assume (18) is false, then, $\check{\phi} \in [-\pi/2, 0]$ if $w(\pi/2) > w(0)$ and $\check{\phi} \in [\pi/2, \pi]$ if $w(\pi/2) < w(0)$.

Proof: This follows immediately from the fact that $A \cos(\phi - \angle g_{l,m}) + B$ is affine symmetric⁵ in the intervals $[\angle g_{l,m} - \pi/2, \angle g_{l,m} + \pi/2]$ and $[\angle g_{l,m} + \pi/2, \angle g_{l,m} + 3\pi/2]$ with unique extreme point. \square

Proposition 3 can determine, via 3 points, a $\pi/2$ -length closed-interval in which $\check{\phi} = \angle g_{l,m}$ is the unique local (and therefore global) minimum. It is now possible to invoke a binary search to approximate $\angle g_{l,m}$ by $\hat{\phi}$ where for any accuracy, say $\eta > 0$, it takes an additional $-\log_2(2\eta/\pi)$ points (transmission cycles) to ensure $\check{\phi} \in [\angle g_{l,m} - \eta, \angle g_{l,m} + \eta]$. In exactly the same manner it can be shown that the optimal value of $\hat{\theta}$ in (12) can be approximated η closely using $3 - \log_2(\eta/\pi)$ transmission cycles. The algorithm is summarized in Table I.

⁴Since w is assumed monotone and continuous and its argument is a 2π periodical sinusoid such point always exist if $w(0) \neq w(\pi)$.

⁵A function $f : \mathbb{R} \rightarrow \mathbb{R}$ is affine symmetric if there exist some $a \in \mathbb{R}$ such that $f(x - a) = f(a - x)$.

TABLE I
LINE SEARCH

function: $z = \text{LineSearch}(w, z_{\min}, z_{\max}, \eta)$

- 1) Initialize: $z = z_{\max}/2$; $a, b, c = 0$; $L = z_{\max}$.
- 2) If $|w(z_{\max}/2) - w(0)| > |w(z_{\max}/2) - w(z_{\max})|$, set $a = 1$.
- 3) If $w(0) < w(z_{\max}/2)$, set $b = 1$.
- 4) If $w(z_{\max}) > w(z_{\max}/2)$ set $c = 1$.
- 5) $z_{\max} = z_{\max}a(1-b) + z_{\max}(1-a)c$;
- 6) $z_{\min} = z_{\max} - L$.
- 7) While $(|z_{\max} - z_{\min}| \geq \eta)$
- 8) $z = (z_{\max} + z_{\min})/2$
- 9) If $(w(z_{\min}) \leq w(z_{\max}))$, set $z_{\max} = z$, otherwise, set $z_{\min} = z$.
- 10) end while

end **LineSearch**

IV. BLIND NULL SPACE LEARNING VIA THE CYCLIC JACOBI TECHNIQUE

In this section we present a Blind Null Space Learning (BNSL) algorithm for solving the blind null space learning problem based on Theorems 1- 2 and on the line search algorithm in Table I. The SU is using a pre-coding matrix that is being updated at each learning stage, let \mathbf{W}_k be that pre-coding matrix at stage k .

A. The Blind Null Space Learning (BNSL) Algorithm

Let $\mathbf{U}\Sigma\mathbf{V}^*$ be \mathbf{H} 's Singular Value Decomposition, where \mathbf{V} and \mathbf{U} are $n_t \times n_t$ and $n_r \times n_r$ unitary matrices respectively and assume that $n_t > n_r$. The matrix Σ is an $n_r \times n_t$ diagonal matrix with real nonnegative diagonal entries $\sigma_1, \dots, \sigma_d$ that are arranged as $\sigma_1 \geq \sigma_2 \geq \dots \geq \sigma_d > 0$. We assume with no loss of generality that $n_r = d (= \text{Rank}(\mathbf{H}))$. Note that

$$\mathcal{N}(\mathbf{H}) = \text{span}(\mathbf{v}_{n_r+1}, \dots, \mathbf{v}_{n_t}) \quad (19)$$

where \mathbf{v}_i denotes \mathbf{V} 's i th column. From the SU point of view, it is sufficient to learn $\mathcal{N}(\mathbf{G})$ (which is equal to $\mathcal{N}(\mathbf{H})$ because $\mathbf{G} = \mathbf{V}\Lambda\mathbf{V}^*$ where $\Lambda = \Sigma^T\Sigma$). It is possible to diagonalize \mathbf{G} via the blind Jacobi algorithm given in Table II. The secondary user's initial (i.e. at $k = 0$) pre-coding matrix is $\mathbf{W}_0 = \mathbf{I}$. Each Jacobi step is equivalent to a learning stage (indexed by

k , see Figure 2) and is composed of K transmission cycles where the secondary user obtains $(\hat{\theta}_k, \hat{\phi}_k)$ at its end, and updates its pre-coding matrix

$$\mathbf{W}_k = \mathbf{W}_{k-1} \mathbf{R}_{l_k, m_k}(\hat{\theta}_k, \hat{\phi}_k) \quad (20)$$

Obtaining $(\hat{\theta}_k, \hat{\phi}_k)$ at each learning stage requires two binary searches where the value of $f_k(S(\mathbf{G}, \mathbf{W}_k \mathbf{r}_{l_k, m_k}(\theta(n), \phi(n))))$ for each search point (θ_n, ϕ_n) is obtained by a transmission cycle in which the SU is transmitting

$$\begin{aligned} \mathbf{x}_2(t) = \tilde{\mathbf{x}}(n) = \mathbf{W}_{k-1} \mathbf{r}_{l_k, m_k}(\theta_n, \phi_n) \in \mathbb{C}^{n_t} \\ , \quad \forall (n-1)N \leq t \leq nN \end{aligned} \quad (21)$$

i.e. for each k , the SU performs two one-dimensional binary searches to obtain $(\hat{\theta}_k, \hat{\phi}_k)$ (the minimum of $f_k(S(\mathbf{G}, \mathbf{W}_{k-1} \mathbf{r}_{l_k, m_k}(\theta, \phi)))$ from the set $\{\mathbf{r}_{l_k, m_k}(\theta_n, \phi_n), f_k(S(\mathbf{G}, \mathbf{W}_{k-1} \mathbf{r}_{l_k, m_k}(\theta_n, \phi_n)))\}_{n=(k-1)K}^{kK-1}$. As the algorithm proceeds, $\hat{\theta}_k$ approaches zero (this will be shown on Section V), a fact that can be used for a stopping criteria.

Assume that the BNSL algorithm is performed until $k = k_s$. Then the SU pre-coding matrix \mathbf{T}_{k_s} is given by

$$\mathbf{T}_{k_s} = [\mathbf{w}_{i_1}^{k_s}, \dots, \mathbf{w}_{i_{n_t} - n_r}^{k_s}] \quad (22)$$

where \mathbf{w}_i^k is \mathbf{W}_k 's i th column and i_1, i_2, \dots, i_{n_t} is an indexing such that

$$(\mathbf{w}_{i_1}^{k_s})^* \mathbf{G} \mathbf{w}_{i_1}^s \leq (\mathbf{w}_{i_2}^{k_s})^* \mathbf{G} \mathbf{w}_{i_2}^s \leq \dots \leq (\mathbf{w}_{i_{n_t}}^{k_s})^* \mathbf{G} \mathbf{w}_{i_{n_t}}^s \quad (23)$$

Thus, the interference power that the SU inflict to the PU is bounded

$$\|\mathbf{H}_{12} \mathbf{x}_{12}\|^2 \leq p_2 \|\mathbf{H}_{12} \mathbf{T}_{k_s}\|^2 \quad (24)$$

where p_2 is the SU's transmit power. In Section V it will be shown that $\|\mathbf{H}_{12} \mathbf{T}_k\|^2$ converges quadratically to zero as is bounded by for sufficiently small η and ultimately bounded by $O(\eta^2)$.

It is important to note the eigenvalues of \mathbf{G} cannot be obtained by the BNSL algorithm. This leaves the SU with the problem of how to determine which of the columns of \mathbf{W}_{k_s} corresponds to \mathbf{H}_{12} 's null space, if it doesn't know its rank in advance. The problem however can be solved due to the fact that $\mathbf{v}_i \in \mathcal{N}(\mathbf{G})$ iff $S(\mathbf{G}, \mathbf{v}_i) = S(\mathbf{G}, 2\mathbf{v}_i)$. Denote $h_k : \mathbb{C}^{n_t} \rightarrow \mathbb{R}_+$ as

$$h_k(\mathbf{x}) = f_k(S(\mathbf{G}, \mathbf{x})), \quad (25)$$

thus, by setting $\tilde{\mathbf{x}}(n) = \mathbf{v}_i$ and $\tilde{\mathbf{x}}(n+1) = 2\mathbf{v}_i$ it follows that $\mathbf{v}_i \in \mathcal{N}(\mathbf{G})$ iff $h_k(\tilde{\mathbf{x}}(n)) = h_k(\tilde{\mathbf{x}}(n+1))$. The same approximately applies for \mathbf{W}_{k_s} when k_s is sufficiently large.

TABLE II
THE BLIND NULL SPACE LEARNING (BNSL) ALGORITHM

Given the sequence $\{h_k\}_{k \in \mathbb{N}}$ defined in (25)

function: $\mathbf{W} = \mathbf{BNSL}(\{h_k\}_{k \in \mathbb{N}}, n_t)$

- 1) Initialization: $k = 1$, $\mathbf{W}_0 = \mathbf{I}_{n_t}$, $\Delta_j = 2\xi, \forall j \leq 0$.
- 2) while $(\max_{j \in \{k-n(n-1)/2, \dots, k\}} \Delta_j \geq \xi)$.
- 3) $w(\phi) = h_k(\mathbf{W}_{k-1} \mathbf{r}_{l_k, m_k}(\pi/3, \phi))$.
- 4) set $\hat{\phi}_k = \mathbf{LineSearch}(w, 0, \pi, \eta)$
- 5) $w(\theta) = h_k(\mathbf{W}_{k-1} \mathbf{r}_{l_k, m_k}(\theta, \hat{\phi}_k))$.
- 6) set $\tilde{\theta}_k = \mathbf{LineSearch}(w, 0, \pi/2, \eta)$
- 7) set $\hat{\theta}_k$ according to (13).
- 8) set $\Delta_k = |\hat{\theta}_k|$
- 9) set $\mathbf{W}_k = \mathbf{W}_{k-1} \mathbf{R}_{l_k, m_k}(\hat{\theta}_k, \hat{\phi}_k)$, $\mathbf{W} = \mathbf{W}_k$
- 10) $k = k + 1$.
- 11) end while

B. The Reduced Complexity Blind Null Space Learning (RC-BNSL) Algorithm

In the BNSL algorithm, each step requires two line searches where each search point is obtained by a transmission cycle. These transmission cycles are the dominant latency factor in the learning process since the rest of the calculations are performed off line at the secondary device processing unit. Roughly speaking, the complexity of the cyclic Jacobi technique grows like n_t^2 which is the dimension of the matrix \mathbf{G} (the convergence and complexity will be discussed in Section V). In this Section we present an algorithm that converts the problem of blind null space learning of the $n_t \times n_t$ matrix \mathbf{G} into an equivalent problem of blind null Space learning of $n_t - n_r$ matrices where each is an $n_r \times n_r$ matrix. This is possible due to the fact that $\mathbf{G} = \mathbf{H}^* \mathbf{H}$ is a n_r -rank matrix and therefore has a $n_t - n_r$ dimensional null space⁶. The resulting complexity grows like $(n_t - n_r)n_r^2$, therefore, the algorithm is efficient if n_t is sufficiently larger than n_r (which is a practical case since SU systems will be more sophisticated than the PU).

The idea behind the RC-BNSL algorithm is described in the following observation:

Observation 4: Let $\mathbf{H} \in \mathbb{C}^{n_r \times n_t}$ be an n_r -rank ($n_t > n_r$) matrix and let $\mathbf{U} = [\mathbf{u}_1, \dots, \mathbf{u}_{n_r+1}] \in$

⁶If the matrix \mathbf{G} was known, the fact that $\text{rank}(\mathbf{G}) = n_r$ could have been utilized by QR decomposition (which cannot be done blindly) prior to the diagonalization.

$\mathbb{C}^{n_t \times (n_r+1)}$ be an orthonormal matrix (that is, a matrix whose columns are an orthonormal set i.e. $\mathbf{U}^* \mathbf{U} = \mathbf{I}$) and let $\tilde{\mathbf{H}} = \mathbf{H}\mathbf{U}$. If $\tilde{\mathbf{u}} \in \mathcal{N}(\tilde{\mathbf{H}})$ then $\mathbf{u} = \mathbf{U}\tilde{\mathbf{u}} \in \mathcal{N}(\mathbf{H})$.

Proof: This is due to $\mathbf{H}\mathbf{u} = \mathbf{H}\mathbf{U}\tilde{\mathbf{u}} = \mathbf{0}$ since $\tilde{\mathbf{u}} \in \mathcal{N}(\mathbf{H}\mathbf{U})$.

The RC-BNSL algorithm is carried out as follows: The secondary user begins with an initial pre-coding matrix $\mathbf{U}^{(0)} \in \mathbb{C}^{n_r \times (n_t - n_r)}$ which is composed of the last $n_t - n_r$ columns of some unitary matrix $\mathbf{W} \in \mathbb{C}^{n_t \times n_t}$. Let $\mathbf{H}_{\text{eq}}^{(1)} = \mathbf{H}\mathbf{U}^{(0)} \in \mathbb{C}^{n_r \times (n_r+1)}$, then there exists at least one degree of freedom in this channel. The SU can apply the BNSL algorithm on $\mathbf{G}^{(1)} = \mathbf{H}_{\text{eq}}^{(1)*} \mathbf{H}_{\text{eq}}^{(1)}$ and obtains a pre-coding matrix $\mathbf{U}_k^{(1)}$ such that $\tilde{\mathbf{U}}^{(1)} = \lim_{k \rightarrow \infty} \tilde{\mathbf{U}}_k^{(1)}$ (in Section V it is shown that the limit exists) and that

$$\mathbf{\Lambda}^{(1)} = \tilde{\mathbf{U}}^{(1)*} \mathbf{G}^{(1)} \tilde{\mathbf{U}}^{(1)} = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & \lambda_{n_r}^{(1)} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \lambda_1^{(1)} \end{bmatrix} \quad (26)$$

Now that $\mathbf{G}^{(1)}$ is diagonalized the first degree of freedom is given by $\mathbf{v}^{(1)} = \mathbf{U}^{(0)} \tilde{\mathbf{u}}_1^{(1)}$ where $\tilde{\mathbf{u}}_1^{(1)}$ is the first column of $\tilde{\mathbf{U}}^{(1)}$, (that lies in the null space of $\mathbf{G}^{(1)}$). The SU then can gain an additional degree of freedom by applying the BNSL algorithm on the following $(n_1 + 1)$ equivalent channel

$$\mathbf{H}_{\text{eq}}^{(2)} = \mathbf{H}\mathbf{U}^{(1)} \quad (27)$$

where $\mathbf{U}^{(1)} \in \mathbb{C}^{n_t \times n_r+1}$ is obtained by concatenating the $n_r - 1$ column of the initial unitary matrix \mathbf{W} with the last n_r column of $\tilde{\mathbf{U}}^{(1)}$ multiplied by $\mathbf{U}^{(0)}$, i.e. let $\hat{\mathbf{U}}^{(1)} = [\tilde{\mathbf{u}}_2^{(1)}, \dots, \tilde{\mathbf{u}}_{n_r+1}^{(1)}]$ then $\mathbf{U}^{(1)} = [\mathbf{w}_{n_r-1}, \mathbf{U}^{(0)} \hat{\mathbf{U}}^{(1)}]$. This equivalent channel is then diagonalized using the BNSL algorithm to obtain $\tilde{\mathbf{U}}^{(2)}$. We now have two degrees of freedom given by $\mathbf{v}^{(2)} = \mathbf{U}^{(1)} \tilde{\mathbf{u}}_1^{(2)}$ and $\mathbf{v}^{(1)}$. This process is repeated until all \mathbf{W} column are used. The RC-BNSL algorithm is summarized in Table III.

V. CONVERGENCE

In this section we study the convergence and complexity properties of the BNSL algorithm. Recall that this algorithm is in fact a blind implementation of the cyclic Jacobi technique whose convergence properties have been extensively studied over the last 50 years. However, while the convergence properties of the Cyclic Jacobi technique directly apply to its Blind implementation

TABLE III
THE REDUCED COMPLEXITY BLIND NULL SPACE LEARNING (RC-BNSL) ALGORITHM

Let $\mathbf{W} \in \mathbb{C}^{n_t \times n_t}$ be a unitary matrix.
function: $[\mathbf{v}_1 \cdots \mathbf{v}_{n_t - n_r}] = \text{RC_BNSL}(\{h_k\}_{k \in \mathbb{N}}, n_r, n_t)$
 Initialize $\mathbf{U}^{(0)} = [\mathbf{w}_{n_t - n_r}, \dots, \mathbf{w}_{n_t}]$.
 for $m = 1, \dots, n_t - n_r$
 $a(\mathbf{x}) = h_k(\mathbf{U}^{(m-1)} \mathbf{x})$
 $\tilde{\mathbf{U}}^{(m)} = \text{BNSL}(a, n_r + 1)$
 $\mathbf{v}_m = \mathbf{U}^{(m-1)} \tilde{\mathbf{u}}_1^{(m)}$
 $\tilde{\mathbf{U}}_m = [\tilde{\mathbf{u}}_2^{(m)}, \dots, \tilde{\mathbf{u}}_{n_r+1}^{(m)}]$
 $\mathbf{U}^{(m)} = [\mathbf{w}_{n_t - n_r - m}^{(m)}, \mathbf{U}^{(m-1)} \tilde{\mathbf{U}}^{(m)}]$
 end for

due to Theorem 1, they cannot be applied directly to the BNSL algorithm. This is due to fact that in the latter algorithm the rotation angles θ_k, ϕ_k are approximated via a line search of finite accuracy η for every k (see Table II, lines 4-6) while in the previous, θ_k, ϕ_k , are equivalent (according to Theorem 2) to the rotation angles of the Cyclic Jacobi technique. Moreover, we would like to make this line search accuracy as small as possible (that is, to make η as large possible) in order to reduce the number of transmission cycles. It is therefore very important to understand how η affects the performance of the BNSL algorithm. In this section we will extend the classic convergence results of the Cyclic Jacobi technique to the BNSL algorithm and show what is the required accuracy in the line search that assures convergence and a minimal level of interference inflicted by the SU to the PU.

A. Overview of the previous work on the convergence of the Jacobi technique

The convergence of the Jacobi technique has been studied extensively over the last sixty years. The first convergence proof of the Cyclic Jacobi technique for complex Hermitian matrices was given by Foster and Henrici [10], which proved that if in each step k : 1) The off diagonal entry satisfies $|a_{l_k, m_k}^k| < c_k |a_{l_k, m_k}^{k+1}|$, where $a_{l, m}^k$ is \mathbf{A}_k 's l, m entry (defined in (7)) and $0 \leq c_k \leq b < 1$ where b is independent of k . 2) The rotation angle (in this paper it is θ_k defined in (12)) lies in some close interval $A \in (-\pi/2, \pi/2)$, then the cyclic technique converges to \mathbf{G} 's EVD. These conditions are satisfied by the BNSL algorithm for any line search accuracy for which $\theta_k \neq 0$.

Thus, to guarantee convergence of the BNSL algorithm to \mathbf{G} 's EVD, one should follow the following rule: If $\theta_k = 0$, phase k should be repeated with η being decreased so that $\theta_k \neq 0$. If any further decrease in η does not change $\theta_k \neq 0$, then a_{l_k, m_k}^k is already zero. This however does not indicate of the convergence rate and does not take into considerations that η cannot be infinitely decreased.

Henrici, and Zimmermann [11] proved that the cyclic Jacobi algorithm for real symmetric matrices has a global linear convergence rate⁷ that depends on the matrix size n_t if the rotation angle $\theta_k \in [-\pi/4, \pi/4]$ for every k . Fernando [12] extended this result to complex Hermitian matrices. A very important result is the ultimate quadratic convergence⁸ rate of the Cyclic Jacobi technique that was shown by Henrici [13] for complex Hermitian matrices with well separated eigenvalues and later enhanced by Wilkinson [14]. The most recent and comprehensive result of the quadratic convergence of Cyclic Jacobi technique that includes multiple and clusters of eigenvalues is due to Hari [15]. Once the Cyclic Jacobi algorithm reaches its quadratic convergence rate it takes a very small number of cycles to reach any desirable accuracy, however, there is no rigorous bound on the number of the required cycles to reach that rate. Brent and Luk [16] have argued heuristically that this number is $O(\log(n_t))$ cycles for $n_t \times n_t$ matrices. This seems to be the case in practice [17, page 429] where such a rapid decline is obtained after three to four cycles [18, page 197]. For further reading the reader is referred to [17–19].

B. Global Linear Convergence rate

The global linear convergence of the Cyclic Jacobi technique was derived by Fernando [12]:

Theorem 5 ([12], Theorem 4): Let \mathbf{G} be a finite dimensional $n \times n$ complex Hermitian matrix and P_k denote the norm of the off diagonal upper triangular (or lower triangular) part of \mathbf{A}_k (as defined in (7)) and let $m = n(n-1)/2$ (which is a cycle length). Then

$$P_{k+m} \leq \rho P_k, \rho = \left(1 - 2^{-(n_t-1)(n_t-2)/2}\right)^{1/2} \quad (28)$$

if

$$\max_{v \leq k+m} \{|\theta_v|\} \leq \frac{\pi}{4} \quad (29)$$

⁷A sequence a_n is said to have a linear convergence rate of $0 < \beta < 1$ if $|a_{n+1}| < \beta|a_n|$.

⁸A sequence is said to have a quadratic convergence rate if there exist some $\beta > 0$ such that $|a_{n+1}| < \beta|a_n|^2$.

Although Theorem 5 is the tightest closed form bound on the global convergence rate of the cyclic Jacobi technique, the practical convergence rate is much faster as discussed in Section V-A.

The condition of Theorem 5 is satisfied in the BNSL algorithm as we show in Theorem 2. However, as described in Table II, the angles θ_k and ϕ_k are approximated via a line search. Thus the off diagonal elements are not completely annihilated, i.e. $[\mathbf{A}_k]_{l_k, m_k} \approx 0$ instead of $[\mathbf{A}_k]_{l_k, m_k} = 0$. In the following Theorem we show what is the required line search accuracy that guarantees the convergence rate in (28).

Theorem 6: Let \mathbf{G} be a finite dimensional $n_t \times n_t$ complex Hermitian matrix and P_k denote the norm of the off diagonal upper triangular (or lower triangular) part of $\mathbf{A}_k = \mathbf{W}_{k-1}^* \mathbf{G} \mathbf{W}_{k-1}$ where \mathbf{W}_k is defined in (20) (see also Table II) and let $m = n_t(n_t - 1)/2$. Let η be the accuracy of the line search, then the BNSL algorithm has a globally linear convergence rate that is given

$$P_{k+m}^2 \leq P_k^2 \left(1 - 2^{-(n_t-2)(n_t-1)/2}\right) + (n_t^2 - n_t)(7 + 2\sqrt{2})\eta^2 \|\mathbf{G}\|^2 \quad (30)$$

Proof: See Appendix B.

To demonstrate this result we substitute $\eta = aP_{qm}/\|\mathbf{G}\|$ and obtain

$$P_{k+m} \leq P_k \left(1 - 2^{\frac{1}{2}(2-n_t)(n_t-1)} + \left(7 + 2\sqrt{2}\right) a^2 (n_t^2 - n_t)\right)^{1/2} \quad (31)$$

It follows that for the BNSL to have a linear convergence rate, it is sufficient that the accuracy be at least:

n	3	4	5	6	8
$\eta \times \frac{\ \mathbf{G}\ }{P_k}$	8×10^{-2}	2×10^{-2}	7×10^{-3}	1×10^{-3}	2×10^{-5}

(32)

Note that the linear convergence coefficient in Theorem 6 may be very small for a large number of transmitting antennas n_t . This may be very bad if this bound turns out to be tight, i.e. if ρ is very close to the true convergence coefficient. In the sequel it will be shown that the actual convergence rate of the BNSL is much faster than the bound in Theorem 6.

C. An Asymptotic Quadratic Convergence Rate

So far it has been shown that for a right chose of η , the BNSL algorithm converges and that for sufficiently small η it has at least a global linear convergence rate. In the following theorem and corollary it is shown what is the desired η to guarantee an asymptotic quadratic convergence

rate and what η guarantees that the SU will meet the maximal interference constraint of the PU.

Theorem 7: Let η be the accuracy of the line search, $\{\lambda_l\}_{l=1}^{n_t}$ be \mathbf{G} 's eigenvalues and let

$$3\delta = \min_{\lambda_l \neq \lambda_r} |\lambda_l - \lambda_r| \quad (33)$$

Let P_k be the norm of the off diagonal upper triangular part of $\mathbf{A}_k = \mathbf{W}_{k-1}^* \mathbf{G} \mathbf{W}_{k-1}$ where \mathbf{W}_k is defined in (20) (see also Table II) and let $m = n_t(n_t - 1)/2$. Assume that the BNSL algorithm has reached a stage where $P_k^2 < \delta^2/8$, then

$$\begin{aligned} P_{k+m}^2 \leq & O\left(\left(\frac{P_k^2}{\delta}\right)^2\right) + O\left(\frac{\eta P_k^{3/2}}{\delta}\right) + O\left(\frac{\eta^2 P_k^{1/2}}{\delta}\right) \\ & + 2(2n_t n_r - n_r^2 - n_r) \eta^2 \|\mathbf{G}\|^2 \end{aligned} \quad (34)$$

Proof: See Appendix C.

Theorem 7 shows that to guarantee quadratic convergence rate the accuracy should be much smaller than P_k^2 , that is, let k_0 be an integer such that $P_{k_0}^2 < \delta^2/8$ then

$$P_{k_0+m} \leq O\left(\left(\frac{P_{k_0}}{\sqrt{\delta}}\right)^2\right) \quad (35)$$

if $\eta \ll P_{k_0}^2$. This implies that once P_k becomes very small such that $P_k^2 \ll \eta$, one cannot guarantee that P_{k+m} will be smaller than P_k^2 but only be smaller than $O(\eta)$. A fact that motivates derivation of a bound on the interference power that the SU inflict to the PU as a function of η .

Corollary 8: Let \mathbf{T}_k be the SU's pre-coding matrix defined in (22). Assume that the conditions of Theorem 7 are satisfied, then

$$\|\mathbf{H}_{12} \mathbf{T}_{k+m}\|^2 \leq O\left(\left(\frac{P_k^2}{\delta^{3/2}}\right)^2\right) + O\left(\frac{\eta P_k^{3/2}}{\delta^2}\right) + 2(2n_t n_r - n_r^2 - n_r) \eta^2 \|\mathbf{G}\|^2 / \delta \quad (36)$$

Proof: This is an immediate consequence of Theorem 11 in Appendix C.

Note that the quantity $\|\mathbf{H}_{12} \mathbf{T}_{k+m}\|^2$ is the only one that interest the SU that applies the BNSL algorithm. Furthermore, once $\eta \gg P_k^2$ the dominant term in the (36) will be $O(\eta)$ i.e., the interference power to the PU will approximately satisfy

$$\|\mathbf{H}_{12} \mathbf{T}_{m+k}\|^2 \leq 2(2n_t n_r - n_r^2 - n_r) \eta^2 \|\mathbf{G}\|^2 / \delta \quad (37)$$

This allows choosing η to guarantee a maximum interference level to the PU, an observation that will be very useful in the simulation part. Theorem 7 and Corollary 8 also imply that the line search accuracy need not be constant during the entire BNSL algorithm but can be refined as the algorithm goes on.

The asymptomatic quadratic convergence rate of Theorem 7 and Corollary 8 is determined by $1/\delta$ where 3δ is the minimal gap between \mathbf{G} 's eigenvalues. In addition, the quadratic convergence rate takes effect only after $P_k^2 < \delta/8$. Such a condition implies that if δ is very small, it will take the BNSL many cycles to reach its quadratic convergence rate. This is problematic since MIMO wireless channels may have very close singular values (recall that \mathbf{H}_{12} square singular values are equal to \mathbf{G} 's first n_r eigenvalues). If we were using the optimal Cyclic Jacobi technique (i.e. no errors due to finite line search accuracy) this would not have a practical implications [15] since quadratic decrease in P_k that is independent of δ occurs prior to the phase where $P_k^2 < \delta/8$. In the following theorem we extend this result to the BNSL algorithm.

Theorem 9: Let η be the accuracy of the line search, $\{\lambda_l\}_{l=1}^{n_t}$ be \mathbf{G} 's eigenvalues such that there exist a cluster of eigenvalues, i.e. $\lambda_l = \lambda + \xi_l$, for $l = n_t - v + 1, \dots, n_t$ where $\sum_{l=n_t-v+1}^{n_t} \xi_l = 0$. Assume that the rest of the non equal eigenvalues are well separated, i.e. $\delta_c \gg |\xi_l|$ where

$$3\delta_c = \min(\Lambda_1 \cup \Lambda_2) \quad (38)$$

$$\begin{aligned} \Lambda_1 &= \{|\lambda_l - \lambda_r| : 1 \leq l < r \leq n_t - v, \lambda_l \neq \lambda_r\} \\ \Lambda_2 &= \{|\lambda_l - \lambda| : 1 \leq l \leq n_t - v\} \end{aligned} \quad (39)$$

Then, once the BNSL algorithm reaches a stage such that $2\delta_c\sqrt{\sum_l \xi_l^2} \leq P_k^2 \leq \delta_c^2/8$, and if $\eta \ll P_k^2$, then

$$P_{k+m} \leq O\left(\left(\frac{P_k}{\sqrt{\delta_c}}\right)^2\right) \quad (40)$$

Proof: See Appendix D.

In the presence of very close eigenvectors cluster, i.e. $\sqrt{\sum_l \xi_l^2} \ll \delta_c$, the distance δ_c will be much greater than δ . In that case, a quadratic decrease in P_k will occur even before P_k becomes smaller than $\delta/\sqrt{8}$ but only satisfies $2\delta_c\sqrt{\sum_l \xi_l^2} \leq P_k^2 \leq \delta_c^2/8$. This quadratic decrease brakes down and become slower as P_k^2 becomes smaller than $2\delta_c\sqrt{\sum_l \xi_l^2}$. From a practical point of view, this is not a problem if one is not interested in decreasing P_k^2 more than $2\delta_c\sqrt{\sum_l \xi_l^2}$ which may be very small. Nevertheless, P_k will eventually decrease quadratically as P_k^2 becomes

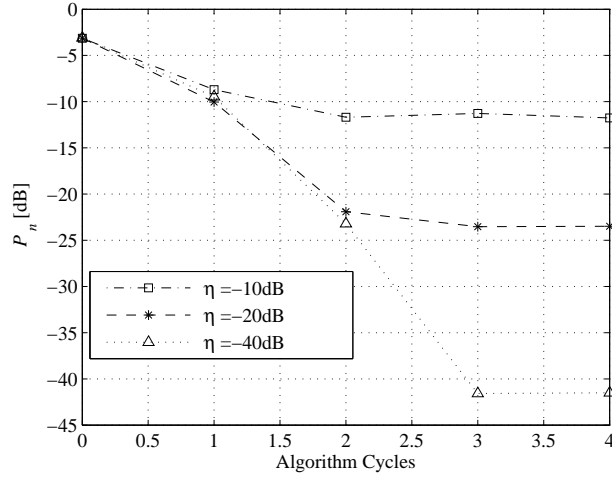


Fig. 4. Simulation results for different line search accuracy values η of the BNSL algorithm for obtaining the null space of \mathbf{H} with $n_t = 3$ transmitting antennas and $n_r = 2$ antennas at the PU receiver. The vertical axis represents the square of the sum of the magnitude of the upper off-diagonal entries of $\mathbf{G} = \mathbf{H}^* \mathbf{H}$ while the horizontal axis represent the number of complete cycles of the BNSL algorithm, i.e. $(n_t^2 - n_t)/2$ learning phases. The matrix \mathbf{G} where normalized such that $\|\mathbf{G}\|^2 = 1$. We used 200 Monte-Carlo trails where the entries of \mathbf{H} are i.i.d. complex Gaussian Random variables.

smaller than $\delta/8$ as required by Theorem 7. This phenomena is for the Cyclic Jacobi technique in [15].

VI. SIMULATIONS

Figure 4 presents simulation results of the BNSL algorithm for different levels of line search accuracies. It is shown that for sufficiently small η the algorithm converges quadratically. The quadratic decrease breaks down when the value of P_k becomes as small as an order of magnitude of η . This result is consistent with the bound that is proposed in Theorem 7. In addition, Figure 4 shows that as long as η is smaller or equal to P_k^2 the decrease in P_{k+m} is almost not affected by different line search accuracies as demonstrated by the fact that the decrease is approximately the same for $\eta = -10, -20, -40$ at the first cycle as well as for $\eta = -20, -40$, at the second cycle. The simulation shows that this phenomena, which is consistent with the asymptomatic behavior as indicated by Theorem 7, is also valid before the algorithm reaches its asymptomatic behavior. Figure 5 describe the interference decrease as a function of the transmission cycles for different line search accuracies. The result is consistent with Corollary 8 where the ultimate

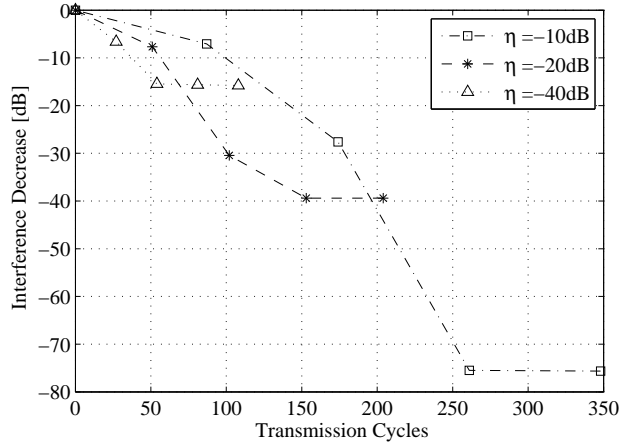


Fig. 5. Simulation results for adaptive line search accuracy values of the BNSL algorithm for obtaining the null space of \mathbf{H} with $n_t = 3$ transmitting antennas and different number of antennas at the PU receiver n_r . The vertical axis represents a bound on the norm of the interference to the PU while the horizontal axis represent the number of transmission cycles. The matrix \mathbf{G} where normalized such that $\|\mathbf{G}\|^2 = 1$. We used 100 Monte-Carlo trails where the entries of \mathbf{H} are i.i.d. complex Gaussian Random variables.

level of the interference is bounded by $O(\eta^2)$. Similar to Figure 4 the interference decrease in Figure 5 is approximately the same for $\eta = -10, -20, -40$ at the first cycle as well as for $\eta = -20, -40$, at the second cycle. Moreover, Figure 5 shows that increasing the values of η reduces the number of transmission cycles drastically as discussed in Section III-C.

Both Figures 4 and 5 suggest that using a low line search accuracy (i.e. larger η) in the first cycle and increasing it from one cycle to the other may reduce the overall transmission cycles with no significant performance loss. This idea is put into practice in Figure 6 where we present simulation results of the interference decrease to the PU as a function of the transmission cycles for an increasing line search accuracy.

VII. CONCLUSIONS

We proposed a blind technique for interference mitigation by secondary cognitive users based on a blind implementation of the well known cyclic Jacobi technique. The only condition required is that the primary user will be using a power control scheme such that for a short time interval, its transmitted power be effected monotonically by the interference from one SU. This includes also a case where there are multiple secondary users in a steady state, i.e. their interference power

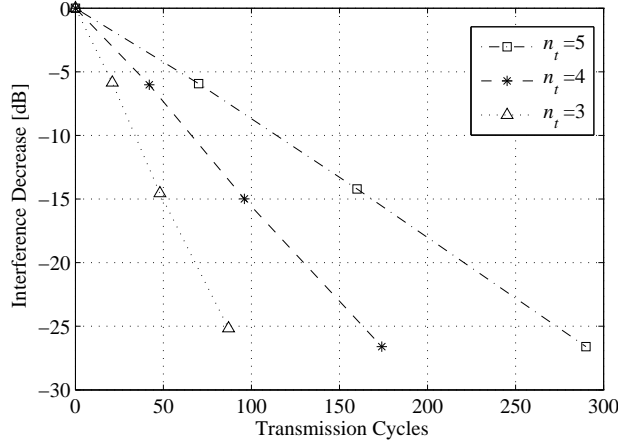


Fig. 6. Simulation results of BNSL algorithm for non constant line search accuracy and different numbers of transmitting antenna at the SU transmitter. The line search accuracy η is -6dB, -8dB -15dB at the first, second and third cycles respectively. The PU has $n_r = 2$ transmitting antennas. The vertical axis represents the reduction in the interference to the PU while the horizontal axis represent the number of transmission cycles. We used 200 Monte-Carlo trails where the entries of \mathbf{H} are i.i.d. complex Gaussian Random variables.

is constant during this time period. The entire learning process is based on energy measurements and on detecting energy variations. This means that the secondary users are not required to be synchronized to the PU pulse time. Furthermore the proposed learning scheme is independent of the PU transmission scheme (i.e. coding, modulation) as long as the power allocation is a monotonous function of the interference from the secondary user.

The Convergence properties of the BNSL algorithm were also explored in this paper. It was shown that the BNSL algorithm maintains the convergence properties of the Cyclic Jacobi technique, that is, a global linear and an asymptotically quadratic convergence rates, as long as the line search accuracy is sufficiently small. It was also shown in simulation that just like in the cyclic Jacobi technique, the BNSL algorithm reaches its quadratic convergence rate in just three to four cycles. Furthermore, we obtained a bound on the interference that the SU inflicts to the PU as a function of the line search accuracy of the BNSL algorithm and provided a mechanism for choosing this line search accuracy to reduce the number of transmission cycles while maintaining low levels of interference to the PU.

It is important to stress that the BNSL algorithm is not necessarily limited to energy measurements taken by the SU and to PU that apply power control. The BNSL algorithm can also be

implemented in other scenarios as long as the SU can learn whether the interference it inflicts to the PU has increased or decreased between one transmission cycle to the other. For example, the secondary user can learn about the interference by observing the PU's modulation scheme since this is too a function of the the interference at the PU. An alternative way for implementing the BNSL algorithm is for the SU to listen to the PU's control channel and extract information about the interference it inflict to the PU.

APPENDIX A

PROOF OF THEOREMS 1 AND 2

The idea behind the proof is that each l, m rotation is equivalent to diagonalizing a 2×2 matrix $\tilde{\mathbf{G}}_{l,m}$ using the rotation matrix $\tilde{\mathbf{R}}(\theta, \phi)$ as follows

$$\tilde{\mathbf{G}}_{l,m} = \begin{bmatrix} g_{l,l} & g_{l,m} \\ g_{m,l} & g_{m,m} \end{bmatrix}, \quad \tilde{\mathbf{R}}(\theta, \phi) = \begin{bmatrix} \cos(\theta) & e^{-i\phi} \sin(\theta) \\ -e^{i\phi} \sin(\theta) & \cos(\theta) \end{bmatrix} \quad (41)$$

Let $\tilde{\mathbf{r}}_1(\theta, \phi), \tilde{\mathbf{r}}_2(\theta, \phi)$ be $\tilde{\mathbf{R}}(\theta, \phi)$'s first and second columns respectively. It follows that

$$[\mathbf{R}_{l,m}(\theta, \phi) \mathbf{G} \mathbf{R}_{l,m}^*(\theta, \phi)]_{l,l} = \mathbf{r}_{l,m}^*(\theta, \phi) \mathbf{G} \mathbf{r}_{l,m}(\theta, \phi) = \tilde{\mathbf{r}}_1^*(\theta, \phi) \tilde{\mathbf{G}}_{l,m} \tilde{\mathbf{r}}_1(\theta, \phi) \quad (42)$$

The first part of the Theorem 1 follows directly from the Rayleigh-Ritz Theorem [see e.g. 20, Theorem 4.2.2] that asserts that

$$\lambda_{l,m}^{\min} = \min_{\mathbf{x} \in \tilde{B}} \mathbf{x}^* \tilde{\mathbf{G}}_{l,m} \mathbf{x} \quad (43)$$

where $\tilde{B} = \{\mathbf{x} \in \mathbb{C}^2 : \|\mathbf{x}\| = 1\}$ and $\lambda_{l,m}^{\min}$ is $\tilde{\mathbf{G}}_{l,m}$'s minimal eigenvalue. It follows that

$$\lambda_{l,m}^{\min} = \tilde{\mathbf{r}}_1^*(\hat{\theta}, \hat{\phi}) \tilde{\mathbf{G}}_{l,m} \tilde{\mathbf{r}}_1(\hat{\theta}, \hat{\phi}) \quad (44)$$

where

$$(\hat{\theta}, \hat{\phi}) = \arg \min_{\theta, \phi \in \mathbb{R}} \tilde{\mathbf{r}}_1^*(\theta, \phi) \tilde{\mathbf{G}}_{l,m} \tilde{\mathbf{r}}_1(\theta, \phi) \quad (45)$$

Equality (44) is due to the fact that for every $\mathbf{x} \in \tilde{B}$ there exist $\theta, \phi \in \mathbb{R}$ such that $\tilde{\mathbf{r}}_1(\theta, \phi) = a\mathbf{x}$ where, $|a| = 1, a \in \mathbb{C}$ and because for every $\mathbf{G} \in \mathbb{C}^{2 \times 2}$ and $\mathbf{x} \in \mathbb{C}^2$ the function S satisfies $S(\mathbf{G}, a\mathbf{x}) = S(\mathbf{G}, \mathbf{x})$ it follows that $\max_{\mathbf{x} \in \tilde{B}} S(\tilde{\mathbf{G}}_{l,m}, \mathbf{x}) \leq \max_{\theta, \phi \in \mathbb{R}} S(\tilde{\mathbf{G}}_{l,m}, \tilde{\mathbf{r}}_1(\theta, \phi))$. In addition, because $\tilde{\mathbf{r}}_1(\theta, \phi) \in \tilde{B}$ for every θ, ϕ we have that $\max_{\mathbf{x} \in \tilde{B}} S(\tilde{\mathbf{G}}_{l,m}, \mathbf{x}) \geq \max_{\theta, \phi \in \mathbb{R}} S(\tilde{\mathbf{G}}_{l,m}, \tilde{\mathbf{r}}_1(\theta, \phi))$ which establishes (44). Note that $\tilde{\mathbf{r}}_1(\hat{\theta}, \hat{\phi})$ is the eigenvector that corresponds $\lambda_{l,m}^{\min}$ and

since $\tilde{\mathbf{r}}_1$ and $\tilde{\mathbf{r}}_2$ are orthonormal, it follows that $\tilde{\mathbf{r}}_2$ is the eigenvector that corresponds to $\tilde{\mathbf{G}}_{l,m}$'s maximal eigenvalue $\lambda_{l,m}^{\max}$. Hence,

$$\tilde{\mathbf{R}}(\hat{\theta}, \hat{\phi}) \tilde{\mathbf{G}}_{l,m} \tilde{\mathbf{R}}^*(\hat{\theta}, \hat{\phi}) = \begin{bmatrix} \lambda_{l,m}^{\min} & 0 \\ 0 & \lambda_{l,m}^{\max} \end{bmatrix} \quad (46)$$

and because $[\mathbf{R}_{l,m}(\hat{\theta}, \hat{\phi}) \mathbf{G} \mathbf{R}_{l,m}^*(\hat{\theta}, \hat{\phi})]_{l,m} = [\tilde{\mathbf{R}}(\hat{\theta}, \hat{\phi}) \tilde{\mathbf{G}}_{l,m} \tilde{\mathbf{R}}^*(\hat{\theta}, \hat{\phi})]_{1,2} = 0$, the desired result follows.

It remains to show the second part of the theorem. The objective function for minimization is

$$S(\mathbf{G}, \mathbf{r}_{l,m}(\theta, \phi)) = [\mathbf{R}_{l,m}(\theta, \phi) \mathbf{G} \mathbf{R}_{l,m}^*(\theta, \phi)]_{l,l} = \cos^2(\theta) |g_{l,l}| + \sin^2(\theta) |g_{m,m}| - |g_{l,m}| \sin(2\theta) \cos(\phi + \angle g_{l,m}) \quad (47)$$

which is a continuous and differentiable (of any order) function of θ and ϕ . Recall that $\mathbf{G} \geq 0$, thus g_{ul} are real non-negative numbers and therefore will be written as g_{ul} instead of $|g_{ul}|$. We first assume that $g_{12} \neq 0$. Setting the gradient to zero yields

$$\begin{aligned} 0 &= -2g_{1,2} \cos(2\theta) \cos(\angle g_{l,m} + \phi) + g_{1,1}(-\sin(2\theta)) + g_{2,2} \sin(2\theta) \\ 0 &= g_{1,2} \sin(2\theta) \sin(\angle g_{l,m} + \phi) \end{aligned} \quad (48)$$

and the solution is

$$(\theta, \phi) \in \left\{ \left((-1)^a \theta_0 + \frac{b\pi}{2}, a\pi - \angle g_{l,m} \right), \left(\frac{\pi b}{2}, -\gamma_{12} + \frac{\pi}{2} + \pi a \right) \right\}_{a,b \in \mathbb{Z}} \quad (49)$$

where

$$\theta_0 = \begin{cases} \frac{1}{2} \tan^{-1} \left(\frac{-2|g_{l,m}|}{g_{l,l} - g_{m,m}} \right) & \text{if } g_{ll} \neq g_{mm} \\ \frac{\pi}{4} & \text{if } g_{ll} = g_{mm} \end{cases} \quad (50)$$

We begin with the family of suspected points $((-1)^a \theta_0 + \frac{b\pi}{2}, a\pi - \angle g_{l,m})$. Since $S(\mathbf{G}, \mathbf{r}_1(\theta, \phi)) = S(\mathbf{G}, \mathbf{r}_1(\theta + \pi, \phi + 2\pi))$, $\forall \theta, \phi \in \mathbb{R}$ it is sufficient to investigate the following subset

$$\begin{aligned} (\theta, \phi) &\in \{ \gamma_1 = (\theta_0, -\angle g_{l,m}), \gamma_2 = (\theta_0 + \pi/2, -\angle g_{l,m}) \\ &\quad, \gamma_3 = (-\theta_0, \pi - \angle g_{l,m}), \gamma_4 = (\pi/2 - \theta_0, \pi - \angle g_{l,m}) \} \end{aligned} \quad (51)$$

Furthermore, because $\mathbf{r}_1(\gamma_1) = -\mathbf{r}_1(\gamma_3)$ and $\mathbf{r}_1(\gamma_2) = -\mathbf{r}_1(\gamma_4)$ it is sufficient to check the points γ_1 and γ_2 . To investigate these points we calculate the Hessian of $S(\mathbf{G}, \mathbf{r}_1(\theta, \phi))$ that is given by

$$\begin{aligned} \nabla^2(S)_{\theta,\theta} &= 4 \sin(2\theta) |g_{l,m}| \cos(\phi + \angle g_{l,m}) + 2 \cos(2\theta) (g_{m,m} - g_{l,l}) \\ \nabla^2(S)_{\theta,\phi} &= 2 \cos(2\theta) |g_{l,m}| \sin(\phi + \angle g_{l,m}) \\ \nabla^2(S)_{\phi,\phi} &= \sin(2\theta) |g_{l,m}| \cos(\phi + \angle g_{l,m}) \end{aligned} \quad (52)$$

and its primary minor (determinant)

$$\begin{aligned} \det(\nabla^2(S)) &= |g_{l,m}| (2 |g_{l,m}| (\cos(2\phi + 2\angle g_{l,m}) - \cos(4\theta)) \\ &\quad + \sin(4\theta) (g_{m,m} - g_{l,l}) \cos(\phi + \angle g_{l,m})) \end{aligned} \quad (53)$$

is equal to $4|g_{l,m}|^2$ for $\gamma_j, j = 1, \dots, 4$. Therefore these are local extreme points. In order to determine which is the maximum and which is the minimum we will substitute these points in $\nabla^2(S)_{\theta,\theta}$

$$\nabla^2(S)_{\theta,\theta}(\theta_0, -\angle g_{l,m}) = \begin{cases} -\frac{|\cos(\theta_0)| 2(4g_{1,2}^2 + (g_{1,1} - g_{2,2})^2)}{(g_{1,1} - g_{2,2})} & \text{if } g_{11} \neq g_{22} \\ 4g_{1,2} & \text{otherwise} \end{cases} \quad (54)$$

$$\nabla^2(S)_{\theta,\theta}(\theta_0 + \pi/2, -\angle g_{l,m}) = \begin{cases} \frac{2(g_{1,1} - g_{2,2})}{|\cos \theta_0|} & \text{if } g_{11} \neq g_{22} \\ -4g_{12} & \text{otherwise} \end{cases} \quad (55)$$

Note that (54) and (55) are of opposite signs, therefore, one of which corresponds to a local minimum while the other to a local maximum. It remains to check the family of suspected points $(\frac{\pi b}{2}, -\gamma_{12} + \frac{\pi}{2} + \pi a)$. By substituting it in (53) we see that it is equal to $-4|g_{l,m}|^2$ for every $a, b \in \mathbb{Z}$. It follows that these suspected points are not extreme points. This establishes that for $g_{1,2} \neq 0$ all of the local minimum points (which are infinitely countable) of $S(\mathbf{G}, \mathbf{r}_1(\theta, \phi))$ are equivalent, i.e. for every local minimum points $(\hat{\theta}, \hat{\phi}), (\theta', \phi')$ such that $(\hat{\theta}, \hat{\phi}) \neq (\theta', \phi')$ we have $S(\mathbf{G}, \mathbf{r}_1(\hat{\theta}, \hat{\phi})) = S(\mathbf{G}, \mathbf{r}_1(\theta', \phi'))$. Thus, every local minimum is also a global minimum. In the case where $g_{1,2} = 0$, the target function is

$$S(\mathbf{G}, \mathbf{r}_{l,m}(\theta, \phi)) = \cos^2(\theta) |g_{l,l}| + \sin^2(\theta) |g_{m,m}| \quad (56)$$

which obviously satisfies the conditions of Theorem 1. This establishes the proof of Theorem 1.

To proof Theorem 2 we substitute $\theta = \pi/3$ in (47)

$$S(\mathbf{G}, \mathbf{r}_{l,m}(\pi/3, \phi)) = -\frac{1}{2}\sqrt{3} |g_{l,m}| \cos(\phi + \angle g_{l,m}) + \frac{3|g_{l,l}|}{4} + \frac{|g_{m,m}|}{4} \quad (57)$$

It follows that in the blind one-dimensional minimization $\min_{\phi} S(\mathbf{G}, \mathbf{r}_{l,m}(\pi/3, \phi))$, $\cos(\tilde{\phi} + \angle g_{l,m}) = 1$ which is obtained by $\tilde{\phi} = -\angle g_{l,m}$ since the line search is carried out in the interval $\phi \in [-\pi, \pi]$. By performing blindly the optimization in (14) we obtain the minimum that corresponds to either γ_1 or γ_2 in (51). Then, the value of $\hat{\theta}$ is chosen such that $\hat{\theta}$ lies in the interval $[-\pi/4, \pi/4]$, thus $(\hat{\theta}, \hat{\phi}) \in \{\gamma_1, \gamma_3\}$ where γ_1, γ_3 are defined in (51). \square

APPENDIX B

PROOF OF THEOREM 6

Consider the first cycle of the cyclic Jacobi technique, i.e. $k = 1, 2, \dots, n_t(n_t - 1)/2$. Denote the number of rotated elements in the l th row by $b_l = n_t - l$ and let

$$\begin{aligned} c_l &= \sum_{j=1}^l b_j = (2n_t - 1 - l)l/2 \\ Z(l, k) &= \sum_{j=1}^{n_t-l} |[\mathbf{A}_k]_{l,j+l}|^2 \\ W(l, k) &= \sum_{j=l+1}^{n_t-1} Z(j, k) \end{aligned} \quad (58)$$

Note that $W(0, k) = P_k^2$. In every cycle, each entry is eliminated once, we therefore denote \mathbf{A}_k 's p, q entry before its annihilation as $g_{q,p}(t)$ where t denotes the number of changes since $k = 0$. After $g_{q,p}(t)$ is annihilated, it will be denoted by $\tilde{g}_{q,p}(\tilde{t})$ where \tilde{t} is the number of changes after the annihilation. The diagonal entries of \mathbf{A}_k will be denoted by x since we are not interested in their values in the course of the proof. This is illustrated in the following example of a 4×4 matrix

$$\begin{aligned} \mathbf{A}_0 = \mathbf{G} & \quad \mathbf{A}_1 \\ \begin{pmatrix} g_{1,1}(0) & g_{1,2}(0) & g_{1,3}(0) & g_{1,4}(0) \\ g_{2,1}(0) & g_{2,2}(0) & g_{2,3}(0) & g_{2,4}(0) \\ g_{3,1}(0) & g_{3,2}(0) & g_{3,3}(0) & g_{3,4}(0) \\ g_{4,1}(0) & g_{4,2}(0) & g_{4,3}(0) & g_{4,4}(0) \end{pmatrix} & \quad \begin{pmatrix} x & \epsilon & g_{1,3}(1) & g_{1,4}(1) \\ \epsilon & x & g_{2,3}(1) & g_{2,4}(1) \\ g_{3,1}(1) & g_{3,2}(1) & x & g_{3,4}(0) \\ g_{4,1}(1) & g_{4,2}(1) & g_{4,3}(0) & x \end{pmatrix} \\ \mathbf{A}_2 & \quad \mathbf{A}_3 \\ \begin{pmatrix} x & \tilde{g}_{1,2}(0) & \epsilon & g_{1,4}(2) \\ \tilde{g}_{2,1}(0) & x & g_{2,3}(2) & g_{2,4}(1) \\ \epsilon & g_{3,2}(2) & x & g_{3,4}(1) \\ g_{4,1}(2) & g_{4,2}(1) & g_{4,3}(1) & x \end{pmatrix} & \quad \begin{pmatrix} x & \tilde{g}_{1,2}(1) & \tilde{g}_{1,3}(0) & \epsilon \\ \tilde{g}_{2,1}(1) & x & g_{2,3}(2) & g_{2,4}(2) \\ \tilde{g}_{3,1}(0) & g_{3,2}(2) & x & g_{3,4}(2) \\ \epsilon & g_{4,2}(2) & g_{4,3}(2) & x \end{pmatrix} \end{aligned} \quad (59)$$

For arbitrary n_t , after the first c_1 sweeps \mathbf{A}_{c_1} 's first column is equal to the following vector:

$$[x, \tilde{g}_{2,1}(n_t - 3), \dots, \tilde{g}_{n_t-1,1}(0), \epsilon]^T \quad (60)$$

and

$$Z(1, c_1) \leq |\tilde{g}_{2,1}(n_t - 3)|^2 + \dots + |\tilde{g}_{n_t-1,1}(0)|^2 + |\epsilon|^2 \quad (61)$$

For $q = 2, \dots, n_t$ we have

$$\begin{aligned}
\tilde{g}_{q,1}(n_t - q - 1) &\leq \cos(\theta_{n_t-1}) \tilde{g}_{q,1}(n - q - 2) - e^{i\phi_{n_t-1}} g_{q,n_t}(1) \sin(\theta_{n_t-1}) \\
&\vdots \\
\tilde{g}_{q,1}(1) &\leq \cos(\theta_{q+1}) \tilde{g}_{q,1}(0) - e^{i\phi_3} g_{q,q+2}(1) \sin(\theta_3) \\
\tilde{g}_{q,1}(0) &\leq |\epsilon| \cos(\theta_q) - e^{i\phi_q} g_{q,q+1}(1) \sin(\theta_q)
\end{aligned} \tag{62}$$

where $\tilde{g}_{q,1}(-1) = \epsilon$. Bounds on $\{\tilde{g}_{q,1}(l)\}_{l=0}^{n_t-q-1}$ can be obtained recursively (i.e. obtaining bound on $\tilde{g}_{q,1}(0)$, substituting and obtaining bound on $\tilde{g}_{q,1}(1)$ and so on...)

$$\begin{aligned}
\tilde{g}_{q,1}(n_t - q - 1) &\leq \epsilon \prod_{v=q}^{n_t-1} \cos(\theta_v) - \sum_{j=q}^{n_t-1} e^{i\phi_j} \sin(\theta_j) g_{q,j+1}(1) \prod_{v=j+1}^{n_t-1} \cos(\theta_v) \\
&= \mathbf{v}(q)^T \mathbf{z}(q) + |\epsilon| \prod_{v=q}^{n_t-1} \cos(\theta_v)
\end{aligned} \tag{63}$$

where $\mathbf{v}, \mathbf{y} \in \mathbb{C}^{n_t-q}$ such that $[\mathbf{v}(q)]_j = -e^{i\phi_{j+1}} g_{q,j+q}(1)$ and $[\mathbf{y}(q)]_j = \sin(\theta_{j+q-1}) \prod_{v=j+q}^{n_t-1} \cos(\theta_v)$.

It follows that

$$\begin{aligned}
|\tilde{g}_{q,1}(n_t - q - 1)|^2 &\leq |\mathbf{y}^T(q) \mathbf{v}(q)|^2 + |\epsilon|^2 \prod_{v=q}^{n_t-1} \cos^2(\theta_v) \\
&\leq \|\mathbf{y}(q)\|^2 \|\mathbf{v}(q)\|^2 + |\epsilon|^2 \prod_{v=q}^{n_t-1} \cos^2(\theta_v)
\end{aligned} \tag{64}$$

Proposition 10:

$$\|\mathbf{y}(q)\|^2 = 1 - \prod_{i=q}^{n-1} \cos(\theta_i) \tag{65}$$

Proof: This is shown by induction. Note that

$$\|\mathbf{y}(q)\|^2 = \sum_{i=q}^{n-1} \sin^2(\theta_i) \prod_{v=i+1}^{n-1} \cos^2(\theta_v) \tag{66}$$

where

$$\prod_{i=l}^m c_i \triangleq 1, \text{ if } l > m. \tag{67}$$

Assume that

$$1 - \prod_{i=q}^{m-1} \cos^2(\theta_i) = \sum_{i=q}^{m-1} \sin^2(\theta_i) \prod_{v=i+1}^{m-1} \cos^2(\theta_v) \tag{68}$$

is true for some $m \in \mathbb{N}$. Then, for $m + 1$ we have

$$\begin{aligned}
&\sum_{i=q}^m \sin^2(\theta_i) \prod_{v=i+1}^m \cos^2(\theta_v) \\
&= \sum_{i=q}^{m-1} \sin^2(\theta_i) \prod_{v=i+1}^m \cos^2(\theta_v) + \sin^2(\theta_m) \prod_{v=m+1}^m \cos^2(\theta_v) \\
&= \cos^2(\theta_m) \sum_{i=q}^{m-1} \sin^2(\theta_i) \prod_{v=i+1}^{m-1} \cos^2(\theta_v) + \sin^2(\theta_m)
\end{aligned} \tag{69}$$

where the last equality is due (67). According to the supposition (68)

$$\begin{aligned} \cos^2(\theta_m) \left(1 - \prod_{i=q}^{m-1} \cos^2(\theta_i) \right) + \sin^2(\theta_m) \\ = 1 - \prod_{i=q}^m \cos^2(\theta_i) \end{aligned} \quad (70)$$

□

By substituting Proposition 10, into (64) one obtains

$$|\tilde{g}_{q,1}(n_t - q - 1)|^2 \leq \underbrace{\left(\sum_{i=1}^{n_t-q} |g_{q,i+q}(1)|^2 \right)}_{=Z(q,c_1)} \left(1 - \prod_{i=c_0+q}^{n_t-1} \cos^2(\theta_i) \right) + |\epsilon|^2 \underbrace{\prod_{v=q}^{n_t-1} \cos^2(\theta_v)}_{\leq 1} \quad (71)$$

thus,

$$|\tilde{g}_{q,1}(n_t - q - 1)|^2 \leq \left(1 - \prod_{i=c_0+q}^{n_t-1} \cos^2(\theta_i) \right) Z(q, c_1) + |\epsilon|^2 \quad (72)$$

and by summing both sides of (72) over $q = 2, \dots, n_t$

$$\begin{aligned} Z(1, c_1) &\leq \sum_{q=2}^{n_t} \left(1 - \prod_{i=c_0+q}^{n_t-1} \cos^2(\theta_i) \right) Z(q, c_1) + (n_t - 1)|\epsilon|^2 \\ &\leq (1 - \prod_{i=c_0+2}^{c_1} \cos^2(\theta_i)) \underbrace{\sum_{q=2}^n Z(q, c_1)}_{W(1, c_1)} + (n_t - 1)|\epsilon|^2 \\ &\leq (1 - \prod_{i=c_0+2}^{c_1} \cos^2(\theta_i)) W(0, 0) + (n_t - 1)|\epsilon|^2 \end{aligned} \quad (73)$$

where the last inequality is due to $P_{c_1} = W(1, c_1) + Z(1, c_1)$, $W(0, 0) = P_0$, and because P_k is a monotonically decreasing sequence ⁹. It follows that

$$Z(1, c_1) = \sin^2(\Psi_{c_0+2, c_1}) W(0, 0) + (n_t - 1)|\epsilon|^2 \quad (74)$$

where

$$\sin^2(\Psi_{c_{l-1}+2, c_l}) = 1 - \prod_{i=c_{l-1}+2}^{c_l} \cos^2(\tilde{\theta}_i) \quad (75)$$

and $|\tilde{\theta}_i| \leq |\theta_i|$. Thus,

$$P_{c_1} = W(1, c_1) + Z(1, c_1) \leq W(0, 0) = P_0 \quad (76)$$

substituting (74) we obtain

$$W(1, c_1) \leq W(0, 0) \cos^2(\Psi_{2, c_1}) - (n_t - 1)|\epsilon|^2 \quad (77)$$

⁹Forsythe and Henrici [10] showed that the sequence P_k is a monotonically decreasing sequence.

Now that this relation is established, it will be applied to \mathbf{A}_{c_1} 's lower $(n_t - 1) \times (n_t - 1)$ block-diagonal. To do that we use the fact that for $J_{c_1+1} = (2, 3), \dots, J_{c_2} = (2, n_t - 1)$ the sum of squares of the first column (which is equal to the first row) remains unchanged, i.e. $\sum_{l=1}^{n_t} |[\mathbf{A}_k]_{1,n}|^2$ is constant for $k = c_1, c_1 + 1, \dots, c_2$. Thus

$$W(l, c_l) \leq W(l-1, c_{l-1}) \cos^2(\Psi_{c_{l-1}+2, c_l}) - (n_t - l)|\epsilon|^2 \quad (78)$$

Continuing this way we obtain

$$W(l, c_l) \leq W(0, 0) \prod_{j=1}^l \cos^2(\Psi_{c_{j-1}+2, c_j}) - \epsilon^2 \sum_{j=1}^l b_j \prod_{v=j+1}^l \cos^2(\Psi_{c_{v-1}+2, c_v}) \quad (79)$$

thus

$$\begin{aligned} Z(l, c_l) &= \sin^2(\Psi_{c_{l-1}+2, c_l}) W(l-1, c_{l-1}) + (n_t - l)|\epsilon|^2 \\ &\leq W(0, 0) \sin^2(\Psi_{c_{l-1}+2, c_l}) \prod_{j=1}^{l-1} \cos^2(\Psi_{c_{j-1}+2, c_j}) \\ &\quad - |\epsilon|^2 \sum_{j=1}^{l-1} b_j \prod_{v=j+1}^{l-1} \cos^2(\Psi_{c_{v-1}+2, c_v}) + (n_t - l)|\epsilon|^2 \end{aligned} \quad (80)$$

After a complete cycle

$$\begin{aligned} P_{c_{n_t-1}}^2 &= \sum_{l=1}^{n_t-2} Z(l, c_{n_t-1}) + |\epsilon|^2 = \sum_{l=1}^{n_t-2} Z(l, c_l) \\ &\leq W(0, 0) \sum_{l=1}^{n_t-2} \sin^2(\Psi_{c_{l-1}+2, c_l}) \prod_{j=1}^{l-1} \cos^2(\Psi_{c_{j-1}+2, c_j}) \\ &\quad - \sum_{l=1}^{n_t-2} |\epsilon|^2 \sum_{j=1}^{l-1} b_j \prod_{v=j+1}^{l-1} \cos^2(\Psi_{c_{v-1}+2, c_v}) + |\epsilon|^2 \sum_{l=1}^{n_t-1} (n_t - l) \end{aligned} \quad (81)$$

Similar to proposition 10, it can be shown that

$$\sum_{l=1}^n \sin^2(\tau_l) \prod_{j=1}^{l-1} \cos^2(\tau_j) = 1 - \prod_{j=1}^n \cos^2(\tau_j) \quad (82)$$

Thus

$$\begin{aligned} P_{c_{n-1}}^2 &\leq W(0, 0) \left(1 - \prod_{j=1}^{n-2} \cos^2(\Psi_{c_{j-1}+2, c_j}) \right) \\ &\quad - \sum_{l=1}^{n-2} \epsilon^2 \sum_{j=1}^{l-1} b_j \prod_{v=j+1}^{l-1} \cos^2(\Psi_{c_{v-1}+2, c_v}) + |\epsilon|^2 \sum_{l=1}^{n-1} b_l \end{aligned} \quad (83)$$

From (75) we have

$$\cos^2(\Psi_{c_{l-1}+2, c_l}) \geq \prod_{v=c_{l-1}+2}^{c_{k_l}} \cos^2(\theta_v) \quad (84)$$

and therefore

$$\begin{aligned} P_{c_{n-1}}^2 &\leq W(0, 0) \left(1 - \prod_{j=1}^{n-2} \prod_{v=c_{j-1}+2}^{c_j} \cos^2(\theta_v) \right) \\ &\quad - \sum_{l=1}^{n-2} |\epsilon|^2 \sum_{j=1}^{l-1} (n-j) \prod_{v=j+1}^{l-1} \prod_{r=c_{v-1}+2}^{c_v} \cos^2(\theta_r) + \frac{|\epsilon|^2(n^2-n)}{2} \end{aligned} \quad (85)$$

Recall that $|\theta_i| < \pi/4$, therefore

$$\begin{aligned} P_{c_{n-1}}^2 &\leq W(0, 0) \left(1 - 2^{-(n-2)(n-1)/2}\right) \\ &\quad - |\epsilon^2| \left(\sum_{l=1}^{n-2} \sum_{j=1}^{l-1} (n-j) 2^{\frac{l^2}{2} - ln + \frac{l}{2} + 9n - 45} - \frac{(n^2-n)}{2}\right) \\ &\leq W(0) \left(1 - 2^{-(n-2)(n-1)/2}\right) + |\epsilon^2| \frac{(n^2-n)}{2} \end{aligned} \quad (86)$$

It remains to relate ϵ to the accuracy of the line search η . Note that the error ϵ in (86) results from the two finite-accuracy (of η accuracy) line-searches in Table II on lines 4 and 6. If η were zero, \mathbf{A}_k 's l, m off diagonal entry would be zero after the k th sweep, i.e.

$$u(\theta^{\text{opt}}, \phi^{\text{opt}}) = 0 \quad (87)$$

where

$$\begin{aligned} u(\theta, \phi) &\triangleq |[\mathbf{R}_{l,m}(\theta, \phi) \mathbf{A}_k \mathbf{R}_{l,m}^*(\theta, \phi)]_{l,m}|^2 = 4(a_{l,m}^k)^2 \sin^2(\gamma_{l,m} + \phi) \\ &\quad + (2 \cos(2\theta) a_{l,m}^k \cos(\angle a_{l,m}^k + \phi) + \sin(2\theta) (a_{l,l}^k - a_{m,m}^k))^2 \end{aligned} \quad (88)$$

and $(\theta_k^{\text{opt}}, \phi_k^{\text{opt}})$ is the value given in Theorem 2 when substituting $\mathbf{G} = \mathbf{A}_k$. Let (θ_k, ϕ_k) be the non optimal value that is obtain by the two line searches (on Line 4 and Line 6), then $|\epsilon|^2 = \max_k u(\theta_k, \phi_k)$. The error $u(\theta_k, \phi_k)$ can be bounded because $\phi_k^{\text{opt}} = \angle a_{l,m}^k$, thus $\phi_k = -\angle a_{l,m}^k + \eta_\phi$ where $|\eta_\phi| < \eta$. Thus

$$u_1(\theta_k, \phi_k) = 4(a_{l,m}^k)^2 \sin^2(\gamma_{l,m} + \phi_k) \leq 4(a_{l,m}^k)^2 \eta^2 \leq 2\|\mathbf{G}\| \eta^2 \quad (89)$$

The second term

$$u_2(\theta_k, \phi_k) = (2a_{l,m}^k \cos(\eta_\phi) \cos^2(2\theta_k) + \sin(2\theta_k) (a_{l,l}^k - a_{m,m}^k))^2 \quad (90)$$

Note that if $a_{ll}^k = a_{mm}^k$, the value $\theta_k = \theta_k^{\text{opt}} \in 0, \pi/4$ since the line search will not miss these points. Now for the case where $a_{ll}^k \neq a_{mm}^k$ we have $\theta_k = \theta_k^s + \eta_\theta$ where

$$\theta_k^s = \frac{1}{2} \tan^{-1}(x_k) \quad (91)$$

and

$$x_k = \frac{2|a_{l,m}^k| \cos(\eta_\phi)}{a_{m,m}^k - a_{l,l}^k} \quad (92)$$

Note that

$$\begin{aligned} u_2(\theta_k, \phi_k) &= (2 \cos(\eta_\phi) a_{l,m}^k (\cos(2\theta_k^s) - 2\eta_\theta \sin(2\theta^*)) \\ &\quad + (a_{l,l}^k - a_{m,m}^k) (\sin(2\theta_k^s) + 2 \cos(2\theta^*) \eta_\theta))^2 \end{aligned} \quad (93)$$

where (θ^*, ϕ^*) is a point on the line that connects the points $(\theta_k^{\text{opt}}, \phi_k^{\text{opt}})$, (θ_k, ϕ_k) . By substituting (91) we obtain

$$u_2(\theta_k \phi_k) = \left(\frac{2 \cos(\eta_\phi) a_{l,m}^k + x_k a_{l,l}^k - x_k a_{m,m}^k}{\sqrt{x_k^2 + 1}} - 4\eta_\theta \sin(2\theta^*) \cos(\eta_\phi) a_{l,m}^k + 2\eta_\theta \cos(2\theta^*) (a_{l,l}^k - a_{m,m}^k) \right)^2 \quad (94)$$

by (92) and by the fact that sinusoidal is bounded by one and by $|\eta_\theta| \leq \eta$ we obtain

$$\begin{aligned} u_2(\theta_k \phi_k) &\leq 4\eta^2 (2|\sin(2\theta^*)| a_{l,m}^k + \cos(2\theta^*) |a_{l,l}^k - a_{m,m}^k|)^2 \\ &\leq 4\eta^2 (4\sin^2(2\theta^*) |a_{l,m}^k|^2 + 2\sin(4\theta^*) |a_{l,m}^k| |a_{l,l}^k - a_{m,m}^k| + \cos^2(2\theta^*) |a_{l,l}^k - a_{m,m}^k|^2) \\ &\leq 4\eta^2 (2|a_{l,m}^k|^2 + 2\sin(4\theta^*) |a_{l,m}^k| |a_{l,l}^k - a_{m,m}^k| + |a_{l,l}^k - a_{m,m}^k|^2 + 2|a_{l,m}^k|^2) \end{aligned} \quad (95)$$

$$u_2(\theta_k, \phi_k) \leq 4\eta^2 (2\|\mathbf{G}\|^2 + \sqrt{2}\|\mathbf{G}\|\|\mathbf{G}\| + \|\mathbf{G}\|^2) \quad (96)$$

Thus

$$|\epsilon|^2 = \max_k u(\theta_k, \phi_k) \leq 2(7 + 2\sqrt{2})\eta^2 \|\mathbf{G}\|^2 \quad (97)$$

This expression is substituted into (86) and the desired result follows.

APPENDIX C

PROOF OF THEOREM 7

We first assume that \mathbf{G} 's eigenvalues are all distinct. From (63) it follows that

$$|\tilde{g}_{q,1}(n_t - q - 1)|^2 \leq \sum_{j=q}^{n_t-1} \sin^2(\theta_j) |g_{q,j+1}(1)|^2 + \epsilon^2 \prod_{v=q}^{n_t-1} \cos^2(\theta_v) \quad (98)$$

Similarly to the derivation of (72) we have

$$\begin{aligned} |\tilde{g}_{q,1}(n_t - q - 1)|^2 &\leq Z(q, c_1) \sum_{j=q}^{n_t-1} \sin^2(\theta_j) + |\epsilon|^2 \\ &\leq Z(q, c_1) \sum_{j=2}^{n_t-1} \sin^2(\theta_j) + |\epsilon|^2 \end{aligned} \quad (99)$$

and by summing both sides of (99) (similarly to the derivation of (73)) over $q = 2, \dots, n_t$ it follows that

$$\begin{aligned} Z(1, c_1) &\leq \left(\sum_{j=2}^{n_t-1} \sin^2(\theta_j) \right) \underbrace{\sum_{q=2}^{n_t} Z(q, c_1)}_{W(1, c_1)} + (n_t - 1)|\epsilon|^2 \\ &\leq \left(\sum_{j=2}^{n_t-1} \sin^2(\theta_j) \right) W(0, 0) + (n_t - 1)|\epsilon|^2 \end{aligned} \quad (100)$$

Now that we established this relation we can apply it to the reduced $n_t - l + 1$ lower block diagonal and obtain

$$Z(l, c_l) \leq \left(\sum_{j=c_l-1+1}^{c_l} \sin^2(\theta_j) \right) W(0) + (n_t - l)|\epsilon|^2 \quad (101)$$

After a complete cycle we have

$$\begin{aligned} P_{c_{n_t-1}}^2 &\leq \sum_{l=1}^{n_t-2} Z(l, c_{n_t-1}) + |\epsilon|^2 = \sum_{l=1}^{n_t-2} Z(l, c_l) + |\epsilon|^2 \\ &\leq W(0, 0) \sum_{j=1}^{n_t(n_t-1)/2} \sin^2(\theta_j) + |\epsilon|^2 \sum_{l=1}^{n_t-1} (n_t - l) \end{aligned} \quad (102)$$

We now relate $\sum_{j=1}^{n_t(n_t-1)/2} \sin^2(\theta_j)$ to $W(0, 0)$. Let $\lambda_1, \dots, \lambda_{n_t}$ be \mathbf{G} 's eigenvalues and let

$$3\delta = \min_{i \neq j} |\lambda_i - \lambda_j| \quad (103)$$

and assume that the algorithm has reached a phase where

$$P_k^2 = W(0, k) < \delta^2/8 \quad (104)$$

Note that $|a_{ll}^k - a_{mm}^k|^2 = |a_{ll}^k - \lambda_l - a_{mm}^k + \lambda_m + \lambda_l - \lambda_m|^2 \geq |\lambda_l - \lambda_m|^2 - |a_{ll}^k - \lambda_l|^2 - |a_{mm}^k - \lambda_m|^2$, furthermore, [13, Theorem 1] we have $|a_{ii}^k - \lambda_i| \leq \delta/2$. Thus,

$$|a_{ll}^k - a_{mm}^k| \geq 2\delta - \delta/2 - \delta/2 = \delta \quad (105)$$

Recall that the optimal rotation angle satisfies $\tan(2\theta_k^{\text{opt}}) = 2|a_{l_k m_k}^k|/|a_{l_k l_k}^k - a_{m_k m_k}^k|$ while the actual the rotation angel is $\theta_k = \theta_k^{\text{opt}} + \eta_\theta$. It follows that

$$\begin{aligned} |\sin^2(\theta_k)| &\leq |\sin^2(\theta_k^{\text{opt}})| + |\eta_\theta \sin(2\theta_k^{\text{opt}})| \leq \frac{1}{4}|2\theta_k|^2 \\ &+ |\eta_\theta| \tan(2\theta_k^{\text{opt}}) \leq \frac{1}{2^2} \tan^2(2\theta_k^{\text{opt}}) + |\eta_\theta| \tan(2\theta_k^{\text{opt}}) \\ &\leq \frac{|a_{l_k m_k}^k|^2}{\delta^2} + 2|\eta_\theta| \frac{|a_{l_k m_k}^k|}{\delta} \leq \frac{|a_{l_k m_k}^k|^2}{\delta^2} + \frac{2|\eta_\theta| \sqrt{W(0, k)}}{\delta} \end{aligned} \quad (106)$$

It follows that

$$\begin{aligned} \sum_{k=1}^{n_t(n_t-1)/2} \sin^2(\theta_k) &\leq \sum_{k=1}^{n_t(n_t-1)/2} \left(\frac{|a_{l_k m_k}^k|^2}{\delta^2} + \frac{2|\eta_\theta| \sqrt{W(0, k)}}{\delta} \right) \\ &= \frac{1}{\delta^2} W(0, k) + \frac{\eta_\theta(n_t^2 - n_t)}{\delta} \sqrt{W(0, k)} \end{aligned} \quad (107)$$

By substituting (107) into (102) one obtains

$$P_{c_{n_t-1}}^2 \leq W(0, 0) \left(\frac{1}{\delta^2} W(0, 0) + \frac{(n_t^2 - n_t)|\eta_\theta|}{\delta} \sqrt{W(0, k)} \right) + \frac{|\epsilon|^2}{2} (n_t^2 - n_t), \quad (108)$$

It remains to relate η_θ to the accuracy of the line search η . As a result of the error on Line 4, η_θ depends on η_ϕ as well. Form the proof of Theorem 1 we know that if an accurate line search were invoked, it would produce $\phi_k = -\angle a_{l, m}^k$. However, due to the finite accuracy η , the line search yields $\phi_k = -\angle a_{l, m}^k + \eta_\phi$, where $|\eta_\phi| \leq \eta$. Thus, θ_k is obtained by optimizing a slightly perturbed version of $w(\theta)$, say $\tilde{w}(\theta)$, due to the substitution of ϕ_k into h_k (See Table II Line 5) i.e.

$$\tilde{w}(\theta) = S(\mathbf{A}_k, \mathbf{r}_{l, m}(\theta, \phi_k)) = h_k(\cos^2(\theta)a_{l, l}^k - \cos(\eta_\phi) \sin(2\theta)a_{l, m}^k + \sin^2(\theta)a_{m, m}^k) \quad (109)$$

We first assume that $a_{ll}^k \neq a_{mm}^k$. If both line searches were accurate, the optimal value of θ would be

$$\theta_k^{\text{opt}} = \frac{1}{2} \tan^{-1}(p_k) \quad (110)$$

where $p_k = \frac{2|a_{l,m}^k|}{a_{m,m}^k - a_{l,l}^k}$. If one takes into consideration the non-optimality of the line-search on Line 4 and ignores the non-optimality of the line search on Line 6 then $\tilde{w}(\theta)$ is given in (109) and the optimal value would be

$$\theta_k^s = \frac{1}{2} \tan^{-1}(p_k \cos(\eta_\phi)) \quad (111)$$

The difference $|\theta_k^{\text{opt}} - \theta_k^s|$ is

$$\begin{aligned} |\theta_k^{\text{opt}} - \theta_k^s| &= \left| \frac{1}{2} \tan^{-1}(p_k \cos(\eta_\phi)) - \frac{1}{2} \tan^{-1}(p_k) \right| \\ &\leq \frac{|\eta_\phi \sin(\eta_\phi^*) p_k|}{\cos^2(\eta_\phi^*) p_k^2 + 1} \leq \eta_\phi^2 \frac{|p_k|}{\cos^2(\eta_\phi) p_k^2 + 1} \end{aligned} \quad (112)$$

where $|\eta_\phi^*| \leq \eta_\phi$. It can be easily that

$$\frac{|p_k|}{\cos^2(\eta_\phi) p_k^2 + 1} \leq \frac{1}{|\cos(\eta_\phi)|} \quad (113)$$

Because $\theta_k = \theta_k^s + \eta_\phi$ and $|\eta_\phi| < \eta$, the accumulated effect of the finite accuracy on both Line 4 and Line 6 is bounded by

$$\eta_\theta \leq \eta + \frac{\eta^2}{|\cos(\eta)|} \quad (114)$$

Assuming that η is sufficiently small, (e.g. $\eta \leq \pi/20$) we obtain

$$\eta_\theta \leq 6\eta/5 \quad (115)$$

By substituting (115) and (97) into (108) it follows that

$$\begin{aligned} P_{c_{n_t-1}}^2 &\leq W(0,0) \left(\frac{1}{\delta^2} W(0,0) + \eta \frac{6(n_t^2 - n_t)}{5\delta} \sqrt{W(0,0)} \right) \\ &+ (10 + 2\sqrt{2})(n_t^2 - n_t) \eta^2 \|\mathbf{G}\|^2 \end{aligned} \quad (116)$$

Thus, as long as η is smaller than $W(0,0)$, the BNSL will have a quadratic convergence rate for \mathbf{G} that does not have multiple eigenvalues, i.e. all eigenvalues are distinct. This is not sufficient since we are interested in a matrix \mathbf{G} $n_t - n_r$ with zero eigenvalues.

To extend the proof to the case where the matrix \mathbf{G} has $n_t - n_r$ zero eigenvalues and n_r distinct eigenvalues we shall use the following theorem:

Theorem 11 ([18] Theorem 9.5.1): Let \mathbf{A} be an $n_t \times n_t$ Hermitian matrix with eigenvalues $\{\lambda_l\}_{l=1}^{n_t}$ that satisfy $\lambda_1 \neq \lambda_2 \neq \dots \neq \lambda_{n_r} \neq \lambda_{n_r+1} = \lambda_{n_r+2} = \dots = \lambda_{n_t} = \lambda$. Consider the following partition:

$$\mathbf{A} = \begin{bmatrix} \mathbf{A}_1 & \mathbf{B} \\ \mathbf{B} & \mathbf{A}_2 \end{bmatrix} \quad (117)$$

where \mathbf{A}_1 is $n_r \times n_r$ and \mathbf{A}_2 is $(n_t - n_r) \times (n_t - n_r)$ and let $\delta' > 0$. If $\|(\mathbf{A}_1 - \lambda \mathbf{I})^{-1}\| < 1/\delta'$, then

$$\|\mathbf{A}_2 - \lambda \mathbf{I}\| \leq \|\mathbf{B}\|^2 / \delta' \quad (118)$$

We now apply Theorem 11 to the BNSL Algorithm. Let $\mathbf{A}_1^k, \mathbf{A}_2^k, \mathbf{B}^k$ be \mathbf{A}_k 's submatrices that correspond to the partition in (117). Recall that in our case, $\lambda = 0$, thus, (105) implies that $\|\mathbf{A}_1^k\| > \delta$. Furthermore, by [20, Corollary 6.3.4], the matrix \mathbf{A}_1^k is invertible, thus $\|(\mathbf{A}_1^k)^{-1}\| \leq 1/\delta$, and from Theorem 11 it follows that

$$\|\mathbf{A}_2^k\| \leq \|\mathbf{B}_k\|^2 / \delta \quad (119)$$

To show how this theorem leads to quadratic convergence we first show that once the BNSL algorithm reaches a stage where (104) is satisfied, the affiliation of the diagonal entries in the upper $n_r \times n_r$ -block remains unchanged, i.e. if \mathbf{A}_k satisfies (104) then

$$\arg \min_{l \in L} |\lambda_l - a_{ll}^k| = \arg \min_{m \in L} |\lambda_m - a_{mm}^{k+1}|, \quad \forall L \subseteq \{1, \dots, n_r\} \quad (120)$$

Note that

$$|a_{l_k, l_k}^k - a_{m_k, m_k}^k|^2 \leq \sin^2(\theta_k) \left(2 \cos(\theta_k) a_{l_k, m_k}^k \cos(\phi_k - \angle a_{l_k, m_k}^k) + \sin(\theta_k) (a_{l_k, l_k}^k - a_{m_k, m_k}^k) \right)^2 \quad (121)$$

and that for every θ_k such that $1 \leq k \leq c_{n_r}$ (106) is satisfied. Thus

$$\begin{aligned} |a_{l_k, l_k}^k - a_{l_k, l_k}^{k+1}|^2 &\leq \sin^2(\theta_k) (a_{l_k, m_k}^k + \sin(\theta_k) (a_{l_k, l_k}^k - a_{m_k, m_k}^k))^2 \\ &\leq \sin^2(\theta_k) \left(a_{l_k, m_k}^k + \left((a_{l_k, m_k}^k)^2 / \delta^2 + 2\eta_\theta a_{l_k, m_k}^k / \delta \right)^{1/2} \delta \right)^2 \\ &\leq \sin^2(\theta_k) (a_{l_k, m_k}^k + ((a_{l_k, m_k}^k)^2 + 2\delta\eta_\theta a_{l_k, m_k}^k)^{1/2})^2 \\ &\leq \sin^2(\theta_k) (a_{l_k, m_k}^k + (\delta^2/4 + 2\delta\eta_\theta\delta/2)^{1/2})^2 \\ &\leq \sin^2(\theta_k) (a_{l_k, m_k}^k + \delta(1/4 + \eta_\theta)^{1/2})^2 \\ &\leq \frac{\delta^2}{4} (1 + 4\eta_\theta) \left(1/2 + \sqrt{1/2 + \eta_\theta} \right)^2 \end{aligned} \quad (122)$$

Note that for $\eta \leq 1/100$ and considering (115) it follows that

$$|a_{l_k, l_k}^k - a_{l_k, l_k}^{k+1}| \leq 0.65\delta \quad (123)$$

which establishes (120). Now that (120) is established, it follows that for every l, m such that $l \neq m$ and $1 \leq l \leq n_r$, we have $|a_{ll}^k - a_{mm}^k| \geq \delta$.

After c_{n_t-r} rotations (102) can be written as

$$\begin{aligned} P_{c_{n_r}} &\leq \sum_{l=1}^{n_t-1} Z(l, c_{n_r}) + |\epsilon^2| \sum_{l=1}^{n_r} (n_t - l) \\ &\leq W(0, 0) \sum_{j=1}^{c_{n_r}} \sin^2(\theta_j) + \sum_{l=n_r+1}^{n_t-1} Z(l, c_{n_r}) + |\epsilon^2| \sum_{l=1}^{n_r} (n_t - l) \end{aligned} \quad (124)$$

Recall that $|\epsilon|^2 \leq \max_k u(\theta_k, \phi_k)$ where $u_1(\theta_k, \phi_k)$ and $u_2(\theta_k, \phi_k)$ are defined in (89) and (90).

From (95) we have

$$u_2(\theta_k, \phi_k) \leq 4\eta^2(4P_k^2 + 4P_k\|\mathbf{G}\| + \|\mathbf{G}\|^2) \quad (125)$$

Because $|a_{ll}^k - a_{mm}^k| \geq \delta$, (107) is satisfied and similarly to (116) we obtain

$$\begin{aligned} P_{c_{n_r}} &\leq W(0, 0) \left(\frac{1}{\delta^2} W(0, 0) + \eta \frac{(n_t^2 - n_r)}{\delta} \sqrt{W(0, k)} \right) \\ &\quad + 2(2n_t n_r - n_r^2 - n_r) \eta^2 (9W(0, 0)/2 + 4\sqrt{W(0, 0)}\|\mathbf{G}\| + \|\mathbf{G}\|^2) \\ &\quad + \sum_{l=n_r+1}^{n_t-1} Z(l, c_{n_r}) \end{aligned} \quad (126)$$

It remains to bound the term $\sum_{l=c_{n_r}+1}^{n_t-1} Z(l, c_{n_r})$. Note that for every θ_k such that $1 \leq k \leq c_{n_r}$, (106) is satisfied. Let $Q = \{(l, m) : 1 \leq l \leq n_r < m \leq n_t\}$. Because a_{ll}^k and a_{mm}^k are located in \mathbf{A}_1^k and \mathbf{A}_2^k respectively, it follows that for every k such that $(l_k, m_k) \in Q$

$$\begin{aligned} |a_{q, m_k}^{k+1}|^2 &\leq |a_{q, m_k}^k|^2 + \sin^2(\theta_k) |a_{l_k, q}^k|^2, \text{ for } n_r < q < m_k \\ |a_{m_k, q}^{k+1}|^2 &\leq |a_{m_k, q}^k|^2 + \sin^2(\theta_k) |a_{l_k, q}^k|^2, \text{ for } m_k < q \leq n_t \end{aligned} \quad (127)$$

and from (106)

$$\begin{aligned} |a_{m_k, q}^{k+1}|^2 &\leq |a_{m_k, q}^k|^2 + \left(\frac{|a_{l_k, m_k}^k|^2}{\delta^2} + 2\eta\theta \frac{|a_{l_k, m_k}^k|}{\delta} \right) |a_{l_k, q}^k|^2 \text{ for } m_k < q \leq n_t \\ |a_{q, m_k}^{k+1}|^2 &\leq |a_{q, m_k}^k|^2 + \left(\frac{|a_{l_k, m_k}^k|^2}{\delta^2} + 2\eta\theta \frac{|a_{l_k, m_k}^k|}{\delta} \right) |a_{l_k, q}^k|^2, \text{ for } n_r < q < m_k \end{aligned} \quad (128)$$

These can be bounded by

$$|a_{m_k, q}^{k+1}|^2, |a_{q, m_k}^{k+1}|^2 \leq W^2(0, 0) \left(1 + \frac{1}{\delta^2} \right) + \frac{2\eta\theta}{\delta} W^{3/2}(0, 0) \quad (129)$$

Thus, for every $k \leq c_{n_r}$

$$\sum_{l=n_r+1}^{n_t-1} Z(l, c_{n_r}) = \sum_{q=n_r+1}^{n_t-1} \sum_{t=q+1}^{n_t} |a_{q, t}^k|^2 \leq O\left(\left(\frac{W(0, 0)}{\delta}\right)^2\right) + O\left(\left(\frac{\eta\theta W^{3/2}(0, 0)}{\delta}\right)\right) \quad (130)$$

This, together with (126) and (115) show

$$P_{c_{n-r}}^2 \leq O\left(\left(\frac{W(0,0)}{\delta}\right)^2\right) + O\left(\left(\frac{\eta W^{3/2}(0,0)}{\delta}\right)\right) + O\left(\left(\frac{\eta^2 W^{1/2}(0,0)}{\delta}\right)\right) + 2(2n_t n_r - n_r^2 - n_r) \eta^2 \|\mathbf{G}\|^2 \quad (131)$$

Since P_k is a decreasing sequence, the desired result follows.

APPENDIX D

PROOF OF THEOREM 9

Let $\mathbf{V}_k \Lambda \mathbf{V}_k^* = \mathbf{A}_k$ be \mathbf{A}_k 's EVD, and let

$$\begin{aligned} \tilde{\mathbf{A}}^k &= \mathbf{V}_k \tilde{\Lambda} \mathbf{V}_k^* \\ \hat{\mathbf{A}}^k &= \mathbf{V}_k \hat{\Lambda} \mathbf{V}_k^* \end{aligned} \quad (132)$$

where

$$\begin{aligned} \tilde{\Lambda} &= \text{diag}(\lambda_1, \dots, \lambda_{n_t-v-r}, \underbrace{0 \cdots 0}_r, \underbrace{\lambda \cdots \lambda}_v) \\ \hat{\Lambda} &= \text{diag}(\underbrace{0 \cdots 0}_{n_t-v}, \xi_1, \dots, \xi_v) \end{aligned} \quad (133)$$

Similarly to the derivation of (105), there exists a permutation such that $|a_{ll}^k - a_{rr}^k| \geq \delta_c$ for all l, m such that $l < m$ and $1 \leq l \leq n_t - v - r$ or $n_t - v - r \leq l \leq n_t - v + 1$ and $l > n_t - v$. For such a permutation, let \mathbf{A}_k be partitioned as

$$\mathbf{A}^k = \begin{bmatrix} \mathbf{A}_{11}^k & \mathbf{A}_{12}^k & \mathbf{A}_{13}^k \\ \mathbf{A}_{21}^k & \mathbf{A}_{22}^k & \mathbf{A}_{23}^k \\ \mathbf{A}_{31}^k & \mathbf{A}_{32}^k & \mathbf{A}_{33}^k \end{bmatrix} \quad (134)$$

where $\mathbf{A}_{22}^k \in \mathbb{C}^{r \times r}$ and $\mathbf{A}_{33}^k \in \mathbb{C}^{v \times v}$. Then by [15, Lemma 2.3] we have that

$$\|\mathbf{A}_{ll}^k\|_{\text{off}} \leq \frac{P^2}{2\delta_c}, \text{ for } l = 2, 3. \quad (135)$$

where $\|\mathbf{A}_{ll}^k\|_{\text{off}}$ represent the sum of squares of \mathbf{A}_{ll}^k 's off diagonal entries. The rest of the proof is identical the proof of Theorem 7 from equation (119) and forward since (119) is satisfied by setting

$$\mathbf{A}_1^k = \mathbf{A}_{11}^k, \mathbf{B}^k = [\mathbf{A}_{12}^k, \mathbf{A}_{13}^k], \mathbf{A}_2^k = \begin{bmatrix} \mathbf{A}_{22}^k & \mathbf{A}_{23}^k \\ \mathbf{A}_{32}^k & \mathbf{A}_{33}^k \end{bmatrix} \quad (136)$$

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